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On Reducible but Indecomposable Representations of the Virasoro Algebra

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Abstract

Motivated by the necessity to include so-called logarithmic operators in conformal field theories (Gurarie, 1993) at values of the central charge belonging to the logarithmic series $c_{1,p} = 1 - 6(p-1)^2/p$, reducible but indecomposable representations of the Virasoro algebra are investigated, where L_0 possesses a nontrivial Jordan decomposition. After studying 'Jordan lowest weight modules', where L_0 acts as a Jordan block on the lowest weight space (we focus on the rank two case), we turn to the more general case of extensions of a lowest weight module by another one, where again L_0 cannot be diagonalized. The moduli space of such 'staggered' modules is determined. Using the structure of the moduli space, very restrictive conditions on submodules of 'Jordan Verma modules' (the generalization of the usual Verma modules) are derived. Furthermore, for any given lowest weight of a Jordan Verma module its 'maximal preserving submodule' (the maximal submodule, such that the quotient module still is a Jordan lowest weight module) is determined. Finally, the representations of the $\mathcal{W}$-algebra $\mathcal{W}(2,3^3)$ at central charge $c=-2$ are investigated yielding a rational logarithmic model.

1 Introduction

Since the early works on 2-dimensional conformal field theory [1], the representation theory of the Virasoro algebra $\mathcal{L}$

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{C}{12}\left(n^3 - n\right)\delta_{n+m,0} \qquad \forall n, m \in \mathbb{Z} \tag{1a}$$

$$[C, L_n] = 0 \qquad \forall n \in \mathbb{Z} \tag{1b}$$

has been largely investigated using standard Lie algebra methods such as lowest weight representations and irreducibility. The embedding structure of lowest weight representations was resolved [13, 14, 15, 18, 19] by close examination of the Kac determinant [31].

Only recently it has been shown that for some values of the central charge (when there are fields with integer spaced dimensions) the existence of fields with logarithmic divergences in their four-point-functions is unavoidable [29]. In fact this is true for the whole series of theories on the edge of the conformal grid, namely if $c = c_{1,q} = 1 - 6(q-1)^2/q$, $q \in \mathbb{N}^{\geq 2}$. Other CFTs exhibiting this logarithmic behaviour are the WZNW model on the supergroup $GL(1,1)$ [40], gravitationally dressed conformal field theories [2] and some critical disordered models [8, 35]. These theories have physical relevance as they are supposed to describe aspects of physical systems such as the fractional quantum Hall effect [24, 35, 36, 43], 2-dimensional polymer systems and random walks [7, 9, 41] or 2-dimensional turbulence [25]. In addition, $c = -2$ also appears in the theory of unifying $\mathcal{W}$-algebras [3, 4, 6]. Logarithmic conformal field theories might also prove important for the description of normalizable zero modes for string backgrounds [11, 12, 34].

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Apart from the logarithmic behaviour of four-point-functions these theories also exhibit a peculiar behaviour concerning their fusion structure: If one defines the action of the Virasoro algebra on the tensor product of two Virasoro representations in an appropriate way (see e.g. [38, 26, 39], also c.f. [37]), starting with the set of ordinary lowest weight representations, one is naturally forced to include representations, where L_0 acts as a nontrivial Jordan cell on the lowest weight space. In fact, representations of this kind were already found in [29]. Therefore we will, after some general considerations, focus on such representations, which are generated by two vectors, on which L_0 acts as a nontrivial Jordan block. We will call these representations *Jordan lowest weight modules* (instead of the language of representations of an algebra from now on we will use the equivalent language of modules over an algebra).
Many of the results in this paper have already appeared in [39]; for a broader background the reader may refer to this reference. The paper is organized as follows: After reviewing the most important facts from the theory of lowest weight representations in section 2, the basic definitions for our treatment of nondiagonal representations are given in section 3. In section 4 we proceed by studying the simplest examples of representations of this kind, the above-mentioned Jordan lowest weight modules. The submodules of Jordan lowest weight modules turn out to belong to an even broader class of modules (we will call them *staggered modules*), which we will study in section 5. Sections 6 and 7 then turn back to Jordan lowest weight modules and the classification of their submodules. In section 8, we will generalize our definitions to $\mathcal{W}$-algebras and study the example of $\mathcal{W}(2,3^3)$ at $c=-2$, which will turn out to be rational in a slightly broadened sense. In section 9 we summarize the achieved results and point out directions for future research.

2 Lowest weight modules revisited

The simplest class of modules of the Virasoro algebra $\mathcal{L}$ is the class of lowest weight modules (LWMs). Though the structure of these modules is well known since many years, we will review the basic facts about them in this section. There are two reasons for this. Firstly, we will present the facts using a notation most suitable for our needs. Secondly, our treatment of more complicated modules will sometimes be analogous to the lowest weight case, which we hope to clarify by first presenting the known facts we will use subsequently. The reader familiar with the theory of LWMs may skip this section and directly turn to section 3 on page 5.
Let $\mathcal{U}$ denote the universal enveloping algebra of $\mathcal{L}$. As usual, let $\mathcal{U}_k$ ("k-th level of $\mathcal{U}$") denote the span of monomials of homogeneous degree k: $\langle L_{i_1}^{n_1}\dots L_{i_p}^{n_p}C^{n_c}; n_i \in \mathbb{N}^0, \sum_m n_m i_m = k\rangle$. Furthermore $\mathcal{U}^\pm \subset \mathcal{U}$ and $\mathcal{U}^0 \subset \mathcal{U}$ will denote the universal enveloping algebras of the subalgebras $\mathcal{L}^\pm := \langle L_k; k \gtrless 0\rangle$ and $\mathcal{L}^0 := \langle L_0, C\rangle$ of the Virasoro algebra. Hence, $\mathcal{U} = \mathcal{U}^-\mathcal{U}^0\mathcal{U}^+$.

Definition 2.1 *A module V of the Virasoro algebra is called* ***lowest weight module (LWM)*** *if it contains a subspace $W \subset V$ such that $\dim W = 1$ and $V = \mathcal{U}^+.W$.*

Definition 2.2 *Let V be an $\mathcal{L}$-module. A vector $v \in V$ is called* ***singular*** *if*

(i) $\forall n \in \mathbb{N} : L_{-n}v = 0$

(ii) $L_0 v = hv;\ h \in \mathbb{C}$

(iii) $Cv = cv;\ c \in \mathbb{C}$

Corollary 2.3 *An $\mathcal{L}$-module V is a lowest weight module if and only if it contains a singular vector $v \in V$ such that $V = \mathcal{U}.v$. The number h in definition 2.2 is then called the* ***lowest weight*** *and v a* ***lowest weight vector*** *of the module.*

Remark 2.4 *The central element $C \in \mathcal{L}$ is, in general, represented by a linear operator. But as it belongs to the center of $\mathcal{L}$, in any irreducible representation this operator must be given by a multiple of the identity operator $C = c\mathbb{1}$ (Schur's lemma). The number c is then called* ***central charge****. In indecomposable representations this is also true as long as C is diagonalizable. In this paper we will not deal with other cases. Therefore, we always think of C as a number. This even becomes necessary, if one considers certain extensions of the Virasoro algebra, the so-called $\mathcal{W}$-algebras, which in general can only be consistently defined for certain values of the central charge.*
As a consequence, we will sometimes be sloppy about the operator C and treat it as a number right from the beginning. The scrupulous reader may then e.g. substitute $\mathcal{U}/\langle C - c\mathbb{1}\rangle$ for $\mathcal{U}$.

Definition 2.5 *A LWM V with lowest weight h and LWV v is called* ***Verma module****, if it has the following universal property: For any LWM W with lowest weight h and LWV w, there is a unique $\mathcal{L}$-homomorphism $V \to W$ mapping v to w.*

Theorem 2.6 *For any given $c, h \in \mathbb{C}$, the Verma module $V(h,c)$ exists and is unique up to $\mathcal{L}$-isomorphism. A base of the module is given by*

$$\left\{ L_{i_1} \dots L_{i_k} v | k \in \mathbb{N}^0; i_1 \geq \dots \geq i_k \right\},$$

where v is the lowest weight vector.

Proof: Uniqueness is clear due to the universal property. The existence is proven by construction: $\mathcal{U}$ itself is an $\mathcal{L}$-module by left multiplication. Let $V := \mathcal{U}/\langle L_{-n}, (L_0 - h\mathbb{1}), (C - c\mathbb{1})\rangle$. This obviously is a lowest weight module with lowest weight h and LWV $[\mathbb{1}]$. It is a Verma module by the universal property of $\mathcal{U}$. The last assertion follows from the Poincaré-Birkhoff-Witt theorem for $\mathcal{U}$. ✓

Due to the universal property any lowest weight module is (up to $\mathcal{L}$-isomorphism) a quotient of a Verma module by a proper submodule. We immediately deduce the following

Corollary 2.7 *For any $h, c \in \mathbb{C}$, there is an (up to isomorphism) uniquely determined* ***irreducible*** *or* ***minimal*** *lowest weight module $M(h,c)$. It is given by the quotient of $V(h,c)$ by its maximal proper submodule.*

It is a well known fact [31, 14], that any submodule of a Verma module is generated by singular vectors and therefore is the sum of lowest weight modules. This immediately leads to the question, which Verma modules can be embedded into a given Verma module. This question may be answered using the so-called Shapovalov form which we will define below:
Given a Verma module $V(h,c)$ with LWV v one can define a representation of $\mathcal{L}$ (and thereby of $\mathcal{U}$) on its graded dual $V(h,c)^*$ by setting

$$(L_{i_1}^{n_1} \dots L_{i_p}^{n_p} C^{n_c})^\dagger := C^{n_c} L_{-i_p}^{n_1} \dots L_{-i_1}^{n_p} \tag{2}$$

and

$$((L_{i_1}^{n_1} \dots L_{i_p}^{n_p} C^{n_c})\phi)(w) := \phi((L_{i_1}^{n_1} \dots L_{i_p}^{n_p} C^{n_c})^\dagger w) \tag{3}$$

where $\phi \in V(h,c)^*$ and $w \in V(h,c)$. Let $V^\dagger(h,c) := \mathcal{U}.v^* \subset V(h,c)^*$ denote the **dual module** of $V(h,c)$. It obviously is a lowest weight module with lowest weight h and LWV v^*. The $\mathcal{L}$-homomorphism

$$\begin{cases} V(h,c) & \to & V^\dagger(h,c) \\ u.v & \mapsto & u.v^* \end{cases}$$

together with the natural pairing of $V(h,c)$ with $V(h,c)^*$ then yields a bilinear form $\langle .,. \rangle$ on $V(h,c)$, the **Shapovalov form**. As one easily checks by direct computation, the Shapovalov form is symmetric and obeys

$$k \neq l \Rightarrow \langle V(h,c)_k, V(h,c)_l \rangle = 0 \tag{4}$$

where $V(h,c)_k := \text{eigenspace}(L_0, h+k)$ is the k-th **level** of $V(h,c)$.
One easily sees that the radical of $\langle .,. \rangle$ is exactly given by the maximal proper submodule of $V(h,c)$, and therefore $V^\dagger(h,c) = M(h,c)$. This fact also allows one to define the Shapovalov form on any LWM.
Because of equation (4) it makes sense to examine the determinant of the restriction $\langle .,. \rangle_k$ of the Shapovalov form to the k-th level of a given Verma module. A nontrivial intersection of the k-th level with the maximal proper submodule may then be detected by the vanishing of the corresponding determinant. V. Kac [31] gave an explicit formula for this determinant, which was proven by B.L. Feigin and D.B. Fuks [13]:

Theorem 2.8 *The determinant* $\det_n(h,c)$ *of the matrix of* $\langle .,. \rangle_n$ *on* $V(h,c)_n$ *is given by*

$$\begin{aligned} det_n(h,c) &= K_n \prod_{\substack{r,s \in \mathbb{N} \\ rs \leq n}} (h - h_{c;r,s})^{\mathrm{p}(n-rs)}, \\ h_{c;r,s} &= \tfrac{1}{48}\left((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}(r^2-s^2) - 24rs - 2 + 2c\right), \end{aligned} \tag{5}$$

where $\mathrm{p}(n)$ *denotes the number of* ***partitions*** *of* n *with generating function*

$$\left(\prod_{n \in \mathbb{N}} (1-q^n)\right)^{-1} = \sum_{n \in \mathbb{N}^0} \mathrm{p}(n) q^n, \tag{6}$$

and K_n *are nonvanishing constants (depending on the choice of base).*

By careful examination of this formula B.L. Feigin and D.B. Fuks were able to determine any Verma module that can be embedded in a given one [14, 15]. To this end one parametrizes the central charge by

$$c = 1 - 24k, \tag{7}$$

which leads to

$$h_{r,s} = -k + \tfrac{1}{4}\left((2k+1)(r^2+s^2) + 2\sqrt{k(k+1)}(r^2-s^2) - 2rs\right) \tag{8}$$

for the weights. Evidently, if $\nexists r,s : h = h_{r,s}$, the Verma module $V(h,c)$ itself is irreducible. Using the convention $V_{r,s} := V(h_{r,s}, c)$, the other, so-called **degenerate** cases can be classified as follows:

Theorem 2.9 *Every degenerate representations of* $\mathcal{L}$ *belongs to one of the following classes as determined by* k, $k' := \sqrt{k(k+1)}$:

(i) $k, k' \in \mathbb{Q}$. *In this case* k *must be of the form* $\frac{(p-q)^2}{4pq}$ *with* $p, q \in \mathbb{N}$ *coprime, and therefore* $c = 1 - 6\frac{(p-q)^2}{pq}$. *In addition, one has* $h_{r,s} \in \mathbb{Q}\ \forall r,s \in \mathbb{Z}$. *One distinguishes between three subcases:*

- $q > p > 1$ *(minimal models). We have* $h_{r,s} = \frac{(pr-qs)^2-(p-q)^2}{4pq}$. *Based on the Verma modules* $V_{r,s}$ *with* $1 \leq r \leq q-1$, $1 \leq s \leq p-1$, *one has the following embedding lattices:*

$$\begin{array}{ccccccccc} & \swarrow & V_{-r,s} & \leftarrow & V_{r+2q,s} & \leftarrow & V_{-r-2q,s} & \leftarrow & V_{r+4q,s} \ \cdots \\ V_{r,s} & & & \times & & \times & & \times & \\ & \nwarrow & V_{2q-r,s} & \leftarrow & V_{r-2q,s} & \leftarrow & V_{4q-r,s} & \leftarrow & V_{r-4q,s} \ \cdots \end{array} \tag{9}$$

- *$q > p = 1$ (logarithmic models). Here one has $h_{r,s} = \frac{(r-qs)^2-(q-1)^2}{4q}$. As is readily seen this set is already exhausted by the weights of the form $h_{r,1}$. Based on the Verma modules $V_{r,1}$ with $r \in \{1, \ldots, q-1, q, 2q\}$, we find the following embedding chains:*

$$\begin{array}{ccccccccccl} V_{r,1} & = & V_{2q-r,1} & \leftarrow & V_{2q+r,1} & \leftarrow & V_{4q-r,1} & \leftarrow & V_{4q+r,1} & \cdots & (r \notin q\mathbb{N}) \\ V_{r,1} & \leftarrow & V_{r+2q,1} & \leftarrow & V_{r+4q,1} & \leftarrow & V_{r+6q,1} & \leftarrow & V_{r+8q,1} & \cdots & (r = q, 2q) \end{array} \tag{10}$$

- *$p = q$, i.e. $c = 1$ (Gaussian models). The embedding structure for all degenerate modules is given by $V_{r,s} \leftarrow V_{-r,s}$.*

(ii) $k \in \mathbb{Q}$, $k' \in \mathbb{C}\backslash\mathbb{Q}$ (parabolic models, c.f. [20]). c is still rational; the weights $h_{r,\pm r} \in \mathbb{Q}\ \forall r \in \mathbb{Z}$ are exactly the rational weights. The embedding structure for all degenerate modules is $V_{r,s} \leftarrow V_{-r,s}$.

(iii) $k \in \mathbb{C}\backslash\mathbb{Q}$. Neither c nor the weights (except for $h_{1,1} = 0$) are rational. Again the embedding structure is $V_{r,s} \leftarrow V_{-r,s}$.

3 The general case

For many physical applications, the knowledge of lowest weight modules is completely sufficient. For example, in particle physics all relevant representations must be unitary due to the conservation of probability. Hence, all representations are completely reducible and therefore a direct sum of irreducible representations. With the additional constraint of an energy spectrum bounded from below, irreducible representations are automatically lowest weight (see lemma 3.6 below) and the results of the preceding section are completely satisfactory.
Even in many statistical conformal field theories, where unitarity plays a rather secondary role, one only has to deal with lowest weight representations.
As mentioned before, only recently some cases drew attention, in which this is not true anymore. Even worse, in these cases L_0 is not diagonalizable, but represented by matrices with a nontrivial Jordan decomposition. We therefore must considerably broaden the class of representations we want to deal with. For thermodynamics to make sense we still put some restrictions on the class of representations we want to consider.
In particular, the spectrum of L_0 must be discrete and the real parts must be bounded from below ($\mathrm{tr}e^{-\beta L_0}$ must exist). In fact, in all (mathematically) interesting cases the spectrum will be real.
As a consequence, L_0 possesses a Jordan decomposition $L_0 = L_0^d + L_0^n$ with $[L_0^d, L_0^n] = 0$, where L_0^d is diagonalizable and L_0^n operates nilpotently on its finite dimensional eigenspaces.
For technical reasons we additionally demand C to be diagonalizable.
We will denote the category of Virasoro modules, which meet the above restrictions, by $\mathrm{Mod}_{\mathcal{L}}$. Being a subcategory of the category $\mathrm{Vec}_{\mathbb{C}}$ of complex vector spaces, it clearly is abelian. Its objects will simply be called $\mathcal{L}$-modules.
Though this category is rather large compared to the category of lowest weight modules, the situation is not as bad as one would expect at first sight. Many of the properties of the reducible but indecomposable representations we now have to deal with can be played back to the properties of lowest weight modules. The rest of this section is devoted to this aim.

3.1 Gradation and filtration by $\mathbf{L_0}$

One easily computes

$$(L_0 - h - k)^m L_k = L_k (L_0 - h)^m, \tag{11}$$

and therefore

$$[L_0^d, L_k] = kL_k, \ [L_0^n, L_k] = 0. \tag{12}$$

Now let $V \in \text{Mod}_{\mathcal{L}}$. Clearly one has

$$V = \bigoplus_{h \geq h_{min}} \text{eigenspace}(L_0^d, h). \tag{13}$$

Equation (12) then implies

$$L_k : \text{eigenspace}(L_0^d, h) \to \text{eigenspace}(L_0^d, h+k). \tag{14}$$

For indecomposable V one therefore has

$$V = \bigoplus_{n=0}^{\infty} \text{eigenspace}(L_0^d, h_{min} + n). \tag{15}$$

Definition 3.1 *Let $V_n := \text{eigenspace}(L_0^d, h_{min} + n)$. V_n is called the* **n-*th level*** *of V, V_0 is also called its* ***lowest weight space****.*

Definition 3.2 *L_0^n induces a filtration*

$$\bar{V}^{(1)} \hookrightarrow \bar{V}^{(2)} \hookrightarrow \ldots \hookrightarrow \bar{V}^N = V, \tag{16}$$

where $\bar{V}^{(k)} := \ker((L_0^n)^k)$ and $N \in \mathbb{N} \cup \{\infty\}$. Because of equation (12) the $\bar{V}^{(k)}$ are submodules of V and by definition L_0 operates diagonalizably on the factor modules $V^{(k)} := \bar{V}^{(k+1)}/\bar{V}^{(k)}$. $V^{(k)}$ is called the **k-*th stage*** *of V. The number $N \in \mathbb{N} \cup \{\infty\}$ is called the* ***nilpotency length*** *of V and is denoted by* N-length(V).

For N to be finite it is sufficient (but not necessary) that V is finitely generated as a $\mathcal{U}$-module.

Definition 3.3 *Let $V \in \text{Mod}_{\mathcal{L}}$ be an $\mathcal{L}$-module and $I \subset V$ a submodule. I is called* ***preserving*** *(the nilpotency length), if* N-length$(V/I) =$ N-length(V). *I is called* ***maximal*** *preserving submodule, if there is no preserving submodule $J \subset V$ with $I \subsetneq J$. The module V is called* ***minimal****, if it contains no nonzero preserving submodules.*

3.2 More filtrations

Apart from the filtration (16) we want to introduce two more filtrations. For every $\mathcal{L}$-module $V \in \text{Mod}_{\mathcal{L}}$ one has a chain of embeddings

$$V := V^0 \hookleftarrow V^1 \hookleftarrow V^2 \hookleftarrow \ldots, \tag{17}$$

where V^{k+1} is a maximal proper submodule of V^k. The factor modules $M^k := V^k/V^{k+1}$ are irreducible.

Definition 3.4 *If the chain (17) ends, i.e. $\exists n \in \mathbb{N} : V^n = 0$, then the smallest $n \in \mathbb{N}$ with this property is called the* ***length*** *of the module and is denoted by* length(V). *Otherwise* length$(V) := \infty$.

Lemma 3.5 *For any $\mathcal{L}$-module $V \in \text{Mod}_{\mathcal{L}}$, there is a submodule $W \subset V$, which is a lowest weight module.*

Proof: Without loss of generality let V be indecomposable. Let V_0 denote its lowest weight space. There is at least one $v \in V_0$ which is an eigenvector of L_0. Therefore $\mathcal{U}.v \subset V$ is a lowest weight module. ✓

Corollary 3.6 *Any irreducible $\mathcal{L}$-module $M \in \mathrm{Mod}_{\mathcal{L}}$ is a lowest weight module.*

As even the length of many Verma modules is ∞, we need another, somewhat "coarser" measure of the complexity of an $\mathcal{L}$-module. To this end we use lemma 3.5 and examine sequences of the form

$$V =: W^0 \twoheadrightarrow W^1 \twoheadrightarrow W^2 \twoheadrightarrow \ldots, \tag{18}$$

where $W^k = W^{k-1}/V^{k-1}$ and the $V^k \subset W^k$ are LWMs. Of course, in general there are arbitrarily many sequences of this form, but nevertheless we can define:

Definition 3.7 *The smallest $n \in \mathbb{N}$, for which there is a sequence of the form (18) with $W^n = 0$, is called **lowest weight length** of V and is denoted by $\text{LW-length}(V)$. If there is no such $n \in \mathbb{N}$ we set $\text{LW-length}(V) := \infty$.*

Corollary 3.8 *The lowest weight length of an $\mathcal{L}$-module V is the smallest integer n such that V contains a subspace $W \subset V$ with $\dim W = n$ and $V = \mathcal{U}^+ W$.*

Lemma 3.9 *For any $V \in \mathrm{Mod}_{\mathcal{L}}$ one has*

$$\text{N-length}(V) \leq \text{LW-length}(V) \leq \text{length}(V).$$

Proof: If $\text{length}(V) < \infty$, then any sequence

$$V = V^0 \hookleftarrow V^1 \hookleftarrow \ldots \hookleftarrow V^n = 0$$

as in (17) induces a sequence

$$V = V/V^n \twoheadrightarrow V/V^{n-1} \twoheadrightarrow \ldots \twoheadrightarrow V/V^1 \twoheadrightarrow V/V^0 = 0$$

and $V/V^k = (V/V^{k+1})/(V^k/V^{k+1})$. V^k/V^{k+1} is irreducible and therefore according to corollary 3.6 a LWM, which proves the second inequality. Now consider a sequence

$$V = W^0 \twoheadrightarrow W^1 \twoheadrightarrow W^2 \twoheadrightarrow \ldots$$

as in (18). As the nilpotency length of an LWM always is 1, one either has $\text{N-length}(W^k) = \text{N-length}(W^{k-1})$ or $\text{N-length}(W^k) = \text{N-length}(W^{k-1}) - 1$.
This proves the first inequality. ✓

For future convenience we will now name the simplest cases:

Definition 3.10 *An $\mathcal{L}$-module $V \in \mathrm{Mod}_{\mathcal{L}}$ with $\text{N-length}(V) = \text{LW-length}(V) = k, k \in \mathbb{N}^{\geq 2}$ is called **staggered module**. The number k is called its **rank**.*
*If a staggered module V contains a subspace $W \subset V$ with $\dim W = 1$ and $V = \mathcal{U}^0\mathcal{U}^+ W$, V is called **Jordan lowest weight module (JLWM)**.*
*If a staggered module V with rank k contains a subspace $W \subset V$ with $\dim W = k$ and $V = \mathcal{U}^+ W$, such that $W = \bigoplus_{n=1}^k W_n$, $\forall n : \dim W_n = 1$, $L_0^d W_n = \lambda_n W_n$ and $n \neq m \Rightarrow \lambda_n \neq \lambda_m$, V is called **strictly staggered**.*

For the rest of this paper we will restrict ourselves to the so-called *logarithmic models* with central charge $c_{1,q}, q \in \mathbb{N}^{\geq 2}$ (see theorem 2.9). There are three reasons for this:

- Firstly, for these theories one has towers of weights with integer spacing, so that following V. Gurarie [29] one has to introduce representations with nilpotency length > 1 in order to guarantee the consistency of OPE and conformal blocks. The necessity to do so can also be seen when calculating the fusion product of two LWMs [27, 39].
- Secondly, theories with these central charges have applications in various fields of physical and mathematical interest as e.g. the fractional quantum hall effect [24, 43], the two-dimensional polymer system and 2D random walks [9, 41, 7], turbulence [25] and the theory of unifying $\mathcal{W}$-algebras [3, 4, 6].
- The third reason is a rather technical one: The comparatively simple embedding structure of Verma modules (equation (10)) as compared to the minimal models significantly simplifies the study of modules with nilpotency length > 1.

4 Jordan lowest weight modules

We first investigate the simplest case of modules with nilpotency length > 1, namely the above defined Jordan lowest weight modules (the first example studied by V. Gurarie in [29] was of this type). Their treatment is largely simplified by the following

Lemma and Definition 4.1 *An $\mathcal{L}$-module M is a JLWM of rank k, if and only if there are k linearly independent vectors $v^{(0)}, \dots, v^{(k-1)} \in M$, such that the following conditions are fulfilled:*

(i) $C.v = c\, v \qquad \forall v \in M$

(ii) $L_0.v^{(n)} = h\, v^{(n)} + v^{(n-1)} \;\; \forall n \in \{1, \dots, k-1\}$ *and* $L_0.v^{(0)} = h\, v^{(0)}$

(iii) $L_{-n}.v^{(m)} = 0 \qquad \forall n \in \mathbb{N}, m \in \{0, \dots, k-1\}$

(iv) $M = \mathcal{U}.v^{(k-1)}$.

h is called ***lowest weight*** *of the module and the $v^{(n)}$ its* ***lowest weight vectors****. If $k = 2$, $v^{(1)}$ is called* ***upper*** *and $v^{(0)}$* ***lower*** *lowest weight vector.*

Proof: If the module M fulfills the above conditions, a subspace W as in definition 3.10 is given by $W := \mathrm{span}(v^{(k-1)})$. Conversely, let M be a JLWM of rank k and let $W \subset M$ be a one-dimensional subspace as in definition 3.10. Then choose $0 \neq v^{(k-1)} \in W$. Further let $\forall n \in \{k-2, \dots, 0\} : v^{(n)} := L_0^n v^{(n+1)}$. ✓

For simplicity we will further restrict ourselves to the rank 2 case. Nevertheless, most of the results are analogously valid for higher ranks. The necessary modifications are almost always obvious.
In analogy to the lowest weight case and define:

Definition 4.2 *A Jordan lowest weight module V with lowest weight h and lowest weight vectors $v^{(0)}, v^{(1)}$ is called* ***Jordan Verma module (JVM)****, if it fulfills the following universal property: For any* JLWM *W with lowest weight h and lowest weight vectors $w^{(0)}, w^{(1)}$, there exists a unique $\mathcal{L}$-homomorphism $V \to W$ mapping $v^{(0)}$ to $w^{(0)}$ and $v^{(1)}$ to $w^{(1)}$.*

Theorem 4.3 *For any given $h, c \in \mathbb{C}$, the Jordan Verma module $\tilde{V}(h, c)$ exists and is uniquely determined up to $\mathcal{L}$-isomorphism.*

Proof: As before, uniqueness is clear due to the universal property. Again the existence is proven by construction:

$\hat{V} := \mathcal{U} \times \mathcal{U}$ is an $\mathcal{L}$-module by left multiplication. Let $M \subset \hat{V}$ denote the left ideal generated by $\{(L_0 - h, -1), (0, L_0 - h), (0, L_{-n}), (L_{-n}, 0); n \in \mathbb{N}\}$. Evidently, $\tilde{V}(h,c) := \hat{V}/M$ is the wanted JVM (with lowest weight vectors $[(\mathbb{1}, 0)]$ and $[(0, \mathbb{1})]$). ✓

Remark 4.4 *Alternatively one could have divided $\mathcal{U}$ by the left ideal $\langle (L_0 - h)^2, L_{-n}; n \in \mathbb{N} \rangle$, thereby obtaining lowest weight vectors $[L_0 - h]$ and $[\mathbb{1}]$. Equivalence is easily proven using the Poincaré-Birkhoff-Witt theorem.*

Definition 4.5 *Let V be a JLWM of rank 2. Then $V^{(0)} := \ker L_0^n$ is called **lower** and $V^{(1)} := V/V^{(0)}$ **upper module** of V.*

Corollary 4.6 *Let $V, V^{(0)}$ and $V^{(1)}$ be as above. In $\mathrm{Vec}_{\mathbb{C}}$ one has $V = V^{(0)} \oplus V^{(1)}$. If V is a JVM, then $V^{(0)}$ and $V^{(1)}$ are Verma modules.*

Proof: See the proof of theorem 4.3. ✓

Corollary 4.7 *Let $\tilde{V}(h,c)$ be the Jordan Verma module with lowest weight h and lowest weight vectors $v^{(0)}$ and $v^{(1)}$. Then a base of $\tilde{V}(h,c)$ is given by*

$$\left\{ L_{k_n} \ldots L_{k_1}.v^{(j)}; n \in \mathbb{N}^0, j \in \{0,1\}, 0 < k_1 \leq \ldots \leq k_n \right\}. \tag{19}$$

Proof: See corollary 4.6 and theorem 2.6. ✓

4.1 Shapovalov form and minimal JLWMs

Let $V = \tilde{V}(h,c)$ be the JVM with lowest weight h, central charge c and lowest weight vectors $v^{(0)}, \ldots, v^{(k-1)}$. As before the graded dual V^* becomes an $\mathcal{L}$-module by setting

$$\left. \begin{array}{rl} (L_n\phi)(w) := & \phi(L_{-n}.w) \\ (C\phi)(w) := & \phi(C.w) \end{array} \right\} \forall \phi \in V^*, v \in V. \tag{20}$$

Let $(v^{(n)})^\dagger.v^{(m)} := \delta_{m,n}$, $(v^{(n)})^\dagger.w := 0 \; \forall w \in V_k, k > 0$. One calculates

$$(L_0.(v^{(n)})^\dagger).v^{(m)} = (v^{(n)})^\dagger.L_0.v^{(m)} = h\delta_{n,m} + \delta_{n,m-1} = (h(v^{(n)})^\dagger + (v^{(n+1)})^\dagger).v^{(m)}. \tag{21}$$

$V^\dagger := \mathcal{U}.(v^{(0)})^\dagger$ is a JLWM with lowest weight h, lowest weight vectors

$$\left\{ v^*_{(n)} := (v^{(k-1-n)})^\dagger;\; n \in \{0, \ldots, k-1\} \right\} \tag{22}$$

and central charge c. Just as before, the $\mathcal{L}$-homomorphism

$$\phi : \left\{ \begin{array}{lll} V & \to & V^\dagger \\ u.v^{(k-1)} & \mapsto & u.v^*_{(k-1)} \quad \forall u \in \mathcal{U}^+ \end{array} \right. \tag{23}$$

together with the natural pairing between V and V^* induces the symmetric Shapovalov form $\langle .,. \rangle$ on V.

If, on the other hand, one starts with the claim for symmetry and contravariance with regard to the involution $L_n \mapsto L_{-n}$, one is led to the same form (there is some freedom of choice which stems from the selection of lowest weight vectors).
The radical of $\langle .,. \rangle$ now obviously is *not* the maximal proper submodule of V ($v^{(0)} \notin Rad(V)$). In fact, the quotient of a JVM with lowest weight h by its maximal proper submodule is comparatively uninteresting, as by lemma 3.6 it is just the irreducible LWM $M(h,c)$. A more interesting analogue to maximal proper submodules and irreducible factor modules is given by maximal preserving submodules and minimal factor modules (definition 3.3).

Lemma 4.8 *Let V be a JLWM with lowest weight h. A submodule $I \subset V$ is preserving, if and only if $I \cap V_0 = 0$.*

Proof: If $I \cap V_0 \neq 0$, it follows that $V^{(0)} \subset I$ and therefore M/I has smaller nilpotency length than V. The other direction is clear. ✓

Corollary 4.9 *The maximal preserving submodule of a JVM is uniquely determined. It is given by the union of all preserving submodules.*

Corollary 4.10 *The minimal JLWM for given lowest weight h exists and is uniquely determined up to $\mathcal{L}$-isomorphism. It is given by $\tilde{M}(h,c) := \tilde{V}(h,c)/I_{max}$, where $\tilde{V}(h,c)$ is the JVM with lowest weight h and I_{max} its maximal preserving submodule.*

One easily sees, that the radical of the Shapovalov form on a given JVM is just its maximal preserving submodule. Using the universal property it is clear, that any JLWM with lowest weight h is isomorphic to a factor module $\tilde{V}(h,c)/I$, where I is a preserving submodule of $\tilde{V}(h,c)$. Hence, the Shapovalov form is well defined for any JLWM and its radical always is the maximal preserving submodule. Furthermore, the Shapovalov form is nondegenerate on a minimal JLWM.
Unfortunately the determinant of the Shapovalov form does not prove to be as useful as in the lowest weight case: One easily calculates, that the matrix A_n of the restriction $\langle .,. \rangle_n$ of the Shapovalov form to the n-th level of $\tilde{V}(h,c)$ is given by

$$A_n = \begin{pmatrix} 0 & \tilde{A}_n \\ \tilde{A}_n & * \end{pmatrix}, \tag{24}$$

where $\tilde{A}_n$ is the matrix of the restriction of the Shapovalov form to the n-th level of the Verma module $V(h,c)$. Its determinant therefore is minus the square of the determinant of $\tilde{A}_n$, and its zeroes consequently don't provide any new information about the possible preserving submodules of $\tilde{V}(h,c)$. This was to be expected, since any proper submodule of $\tilde{V}(h,c)^{(0)} = V(h,c)$ is a preserving submodule of $\tilde{V}(h,c)$. In order to determine every possible preserving submodule of $\tilde{V}(h,c)$, we therefore have to use other means than the Shapovalov form.

4.2 Submodules of JVMs

For the sake of simplicity we again restrict ourselves to the case of nilpotency length 2 at central charge $c = c_{1,q}$. Obviously the only interesting cases are the modules $\tilde{V}(h,c)$, where $h = h_{r,s}$ as in theorem 2.9 with $r, s \in \mathbb{N}$:

Lemma 4.11 *Let $\tilde{V}(h,c)$ be the JVM with lowest weight h and $h \neq h_{r,s}\ \forall r,s \in \mathbb{N}$ (see theorem 2.9). Then $\tilde{V}(h,c)$ contains no nonzero proper submodules.*

Proof: Suppose $J \subset \tilde{V}(h,c)$ is a proper submodule. Then $J^{(0)} := J \cap \tilde{V}(h,c)^{(0)}$ and $J^{(1)} := J/J^{(0)}$ are submodules of $\tilde{V}(h,c)^{(0)}$ and $\tilde{V}(h,c)^{(1)}$, respectively. As $h \neq h_{r,s}\ \forall r,s \in \mathbb{N}$, $J^{(0)}$ and $J^{(1)}$ contain no proper submodules and therefore $J = \tilde{V}(h,c)$, which contradicts the assumption of J being a proper submodule. ✓

In order to prevent unnecessary repetition, we will now fix some notations for the rest of this paper:

Definition 4.12 *Let $c = c_{1,q}, q \in \mathbb{N}^{\geq 2}$, and let $V := V(h_{r,s},c)$ be the Verma module with lowest weight $h_{r,s}$, $r,s \in \mathbb{N}$. Then we denote by*

$$V(h_{r,s},c_{1,q}) =: V^1 \hookleftarrow V^2 \hookleftarrow V^3 \hookleftarrow V^4 \ldots$$

the chain of embeddings according to theorem 2.9. Furthermore, let h^k be the lowest weight of V^k and $V^\infty := 0$. In addition, we choose lowest weight vectors $v^k \in V^k$. Then let $\chi_k \in \mathcal{U}^+$, such that $\chi_k.v^k = v^{k+1}$.

We now proceed by searching restrictions which must be met by submodules of a given JVM.

Lemma 4.13 *Let $\tilde{V} = \tilde{V}(h_{r,s},c)$ denote a JVM and other notations as defined above. Let $J \subset \tilde{V}$ be a submodule. As in lemma 4.11 $J^{(0)} := J \cap \tilde{V}^{(0)}$ and $J^{(1)} := J/J^{(0)}$ are submodules of $\tilde{V}^{(0)}$ and $\tilde{V}^{(1)}$ respectively. Then $J^{(1)} = V^n$ and $J^{(0)} = V^m$ with $n,m \in \mathbb{N} \cup \{\infty\}$.*

Proof: Clear. ✓

Lemma 4.14 *Let $\tilde{V}$, J, $J^{(1)} = V^n$ and $J^{(0)} = V^m$ as above. Then $n \geq m$. If J is a proper submodule, then $n > 1$. If J is preserving, then $m > 1$.*

Proof: Let $v^{(1)} + J^{(0)}$, $v^{(1)} \in J$, be a lowest weight vector of $J^{(1)}$. Trivially one has $k < l \Leftrightarrow h^k < h^l$. Therefore $0 \neq (L_0 - h^n).v^{(1)} \in \tilde{V}_{h^n - h^1} \cap J^{(0)} \Rightarrow m \leq n$. The rest is clear. ✓

Even more is true — the submodule J is already completely fixed by the two integers n and m, but due to a lack of notation we postpone this result to the next section (lemma 5.7).

The restrictions imposed by lemma 4.14 on a submodule of a given JVM are necessary, but in general not sufficient for its existence. This becomes clear as one studies the following examples:

Example 4.15 *Let $c = c_{1,q} = 1 - 6\frac{(q-1)^2}{q}$, $q \in \{2,3,\ldots\}$, and $\tilde{V}$ be the JVM with lowest weight 0 and lowest weight vectors $v^{(0)}, v^{(1)}$. Let J denote its maximal preserving submodule. Then $J^{(0)} = V^2$ is the Verma module with lowest weight 1. We will now show, that $J^{(1)} = V^n$ with $n > 2$. Assume, that $J^{(1)} = V^2$ with lowest weight vector $L_1.v^{(1)} + J^{(0)}$. One easily calculates*

$$\begin{aligned} L_{-1}.(L_1.v^{(1)} + \alpha L_1.v^{(0)}) &= [L_{-1}, L_1].(v^{(1)} + \alpha v^{(0)}) \\ &= 2L_0.(v^{(1)} + \alpha v^{(0)}) \\ &= 2v^{(0)}. \end{aligned}$$

Therefore, $J^{(0)} = V^{(0)}$, which contradicts the assumption of J being preserving. ✓

Example 4.16 *Now let $q = 2$ ($c = -2$). We will show that $J^{(1)} = V^3$: Let*

$$w := (2L_2L_1 - L_1^3).v^{(1)} - \tilde{w}, \quad \tilde{w} \in V_3^{(0)}.$$

In $V/J^{(0)}$ one easily calculates ($[w] := w + J^{(0)}$ being the equivalence class of w):

$$L_0.[w] = 3[w].$$

The system of equations

$$L_{-k}.[w] = 0, \qquad k \in \{1, 2, 3\}$$

is uniquely solved by

$$[\tilde{w}] = [L_3.v^{(0)}].$$

In addition,

$$L_{-2}.w = -15L_1.v^{(0)}.$$

Therefore

$$J = \mathcal{U}.w, \quad J^{(0)} = V^2 \quad \textit{and} \quad J^{(1)} = V^3. \checkmark$$

The above examples suggest that for any JVM with lowest weight $h_{r,s}$ there always exists a preserving submodule J with $J^{(0)} = V^2$ and $J^{(1)} = V^3$. Furthermore one might assume that there never is a submodule J with $J^{(0)} = J^{(1)} = V^2$. The first statement will prove to be true, while for the second one we will find counterexamples.

5 Staggered modules

The submodule J in example 4.16 is neither a JLWM nor a lowest weight module, nor is it the direct sum of such. It belongs to the broader class of staggered $\mathcal{L}$-modules defined in definition 3.10, where the nontrivial Jordan decomposition of L_0 shows but at higher levels. In fact, M.R. Gaberdiel and H.G. Kausch [27] found modules of this kind in the fusion product of lowest weight modules at $c = -2$, which do *not* occur as submodules of JLWMs. We therefore leave the submodule point of view and extend our investigations to general modules of this form (we will again restrict ourselves to the rank 2 case). After all, this will also prove useful for the classification of the maximal preserving submodules of JVMs.

Lemma and Definition 5.1 *Let $c = c_{1,q} = 1 - 6\frac{(q-1)^2}{q}$. An $\mathcal{L}$-module M with nilpotency length* N-length$(M) = 2$ *is a staggered module if and only if there is a pair of vectors $v^{(1)}, v^{(0)} \in M$ and numbers $h^{(1)}, h^{(0)} \in \mathbb{C}$, such that the following conditions are met:*

(i) $V^{(0)} := \mathcal{U}.v^{(0)}$ *is LWM with LWV $v^{(0)}$ and lowest weight $h^{(0)}$.*

(ii) $V^{(1)} := M/V^{(0)}$ *is a LWM with LWV $v^{(1)} + V^{(0)}$ and lowest weight $h^{(1)}$.*

(iii) $0 \neq (L_0 - h^{(1)}).v^{(1)} =: v_0 \in V^{(0)}_{h^{(1)}-h^{(0)}}$.

(iv) $L_{-1}.v^{(1)} =: v_1 \in V^{(0)}_{h^{(1)}-h^{(0)}-1}$.

(v) $L_{-2}.v^{(1)} =: v_2 \in V^{(0)}_{h^{(1)}-h^{(0)}-2}$.

*$h^{(1)}$ ($h^{(0)}$) is called **upper (lower) lowest weight** and $v^{(1)}$ ($v^{(0)}$) **upper (lower) lowest weight vector**. The module is strictly staggered, if $h^{(1)} > h^{(0)}$.*

Proof: clear. ✓

Lemma 5.2 *Let M be a staggered module, $h^{(0)}$, $h^{(1)}$, $v^{(0)}$, $v^{(1)}$, v_0, v_1 and v_2 as defined above. Then v_0 is singular.*

Proof: One calculates

$$\begin{aligned} L_{-1}.v_0 &= L_{-1}(L_0 - h^{(1)}).v^{(1)} = (L_0 - h^{(1)}).v_1 + [L_{-1}, L_0].v^{(1)} \\ &= (L_0 - h^{(1)}).v_1 + v_1 = 0. \end{aligned} \tag{25}$$

Analogously one gets $L_{-2}.v_0 = 0$. With

$$\underbrace{[L_{-1}[L_{-1} \dots [L_{-1}}_{n \text{ times}}, L_{-2}]...]] = (-)^n n! L_{-2-n} \tag{26}$$

the assertion follows. ✓

We immediately get the

Corollary 5.3 *For $h^{(1)}, h^{(0)} \notin \{h_{r,s}; r, s \in \mathbb{N}\}$, no proper staggered modules can exist. More precisely: Let V^k and h^k be as in definition 4.12. For given lower lowest weight h^k only staggered modules with upper lowest weight h^m,$m \geq k$, can exist. Of course, at least one such module always exists, namely the corresponding submodule of the JVM $\tilde{V}(h^k, c)$.*

Corollary 5.4 *Let V be a staggered module with lowest weights $h^{(0)}$ and $h^{(1)}$. A submodule $I \subset V$ is preserving, if and only if $I \cap V_{h^{(1)} - h^{(0)}} = 0$. Hence, the maximal preserving submodule of V is uniquely determined.*

Definition 5.5 *A staggered module V is called* ***vermalike****, if it fulfills the following universal property: If M is a staggered module with the same lowest weights and $\phi : M \to V$ is an $\mathcal{L}$-epimorphism, then ϕ is an $\mathcal{L}$-isomorphism.*

Remark 5.6 *Compared to Verma modules and JVMs we defined the universal property in this case "the other way around", as it is true that nonisomorphic vermalike modules always have different $\{v_0, v_1, v_2\}$, but the converse is not true.*

With the above notations, we now return to submodules of JVMs and prove, that the numbers n and m of lemma 4.14 already fix the corresponding submodule completely.

Lemma 5.7 *Let $\tilde{V}$ denote the JVM with lowest weight h with notations as in definition 4.12. Let $I \subset \tilde{V}$ and $J \subset \tilde{V}$ be staggered submodules with $I^{(1)} \cong J^{(1)}$ and $I^{(0)} \cong J^{(0)}$. Then $I = J$.*

Proof: Clearly $I^{(0)} = J^{(0)}$ (theorem 2.9). We assume, that $I \neq J$. Let v_I and v_J be upper LWVs of I and J respectively. Furthermore choose $w = \alpha v_I - v_J$ so that $w + \tilde{V}^{(0)} = 0 + \tilde{V}^{(0)}$ (this is always possible, as theorem 2.9 forces the LWVs of $I^{(1)} \cong J^{(1)} \subset M^{(1)}$ to be scalar multiples of one another). Because of $I \neq J$ we have $w \notin I^{(0)}$ (otherwise, $v_J \in I$). As $v_I + I^{(0)}$ and $v_J + I^{(0)}$ are singular in $M/I^{(0)}$, this also applies to $w + I^{(0)}$. Now $\tilde{V}^{(0)}/I^{(0)}$ contains no nonzero singular vectors on the level in consideration and therefore the assumption must be false. ✓

Remark 5.8 *An analogous statement obviously applies even when $\tilde{V}$ is no JVM but a proper staggered module.*

5.1 Moduli spaces

By definition any staggered module is given as the quotient of a vermalike staggered module by a preserving submodule (a preserving submodule, as before, is a submodule such that the factor module has the same nilpotency length). Therefore we are interested in how many nonisomorphic staggered modules exist for given lowest weights. To this end we will first discuss, which choices of v_0, v_1 and v_2 occur as data of staggered modules.

With the notations from definition 4.12 we want to study all possible vermalike staggered modules with lowest weights $h^{(0)} = h^1$ and $h^{(1)} = h^k$, $k > 1$. Suppose, that there exists a vermalike staggered module M with given data $[v_0], [v_1], [v_2]$, where $v_0, v_1, v_2 \in V^1$ and $[v]$ denotes the image of v under the surjection $V^1 \to M^{(0)}$. Then by the universal properties of $\mathcal{U}$ and V^1 this vermalike staggered module can be constructed as follows:
Let $V := \mathcal{U} \oplus V^1$ where $\mathcal{U}$ is an $\mathcal{L}$-module by left multiplication. Furthermore let $I \in V$ denote the left ideal generated by $\left\{(L_{-1}, -v_1), (L_{-2}, -v_2), (L_0 - h^{(1)}, -v_0)\right\}$, and let $M := V/I$. Evidently, this must be the wanted vermalike staggered module.
Now the question, whether the choices $[v_0], [v_1], [v_2]$ occur as data of a vermalike staggered module, reduces to the question, whether the above constructed module M is a staggered module. In particular, we must have $[v_0] \neq 0$ and we obtain the

Lemma 5.9 *Let M be a vermalike staggered module with lowest weights h^1 and h^k. Then $M^{(0)}$ and $M^{(1)}$ are Verma modules, i.e. $M^{(0)} = V^1$ and $M^{(1)} = V^k$.*

Proof: See the above construction. ✓

We now want to study, under which circumstances M fails to be staggered. We first concentrate on the case $k = 2$ and define $N_2 := h^2 - h^1$. M fails to be staggered if and only if $(0, v^1) \in I$. This is equivalent to

$$\exists \tilde{\xi}, \tilde{\psi} \in \mathcal{U} : \ (\tilde{\xi}.L_{-1} - \tilde{\psi}.L_{-2}, -\tilde{\xi}.v_1 + \tilde{\psi}.v_2) = (0, v^1).$$

If one expands the left hand side of this equation in terms of a PBW base of $\mathcal{U}$, where negative modes are sorted to the right, this turns out to be equivalent to

$$\exists \xi, \psi \in \mathcal{U}^- : \ (\xi.L_{-1} - \psi.L_{-2}, -\xi.v_1 + \psi.v_2) = (0, v^1).$$

Now let $\mathcal{U}^-_n$ be the n-th level of $\mathcal{U}^-$ (i.e. the linear span of monomials $L_{-k_1} \dots L_{-k_m}$ with $\sum_{i=1}^m k_i = n$). Further let $S := \mathcal{U}^-.L_{-1} \cap \mathcal{U}^-.L_{-2} \subset \mathcal{U}^-$. We now want to determine $\dim S_n$ ($S_n := S \cap \mathcal{U}^-_n$).
We first remark, that $\mathcal{U}^-.L_{-1} + \mathcal{U}^-.L_{-2} = \mathcal{U}^-$, which follows from equation (26). Hence,

$$\dim S_n = \dim(\mathcal{U}^-.L_{-1})_n + \dim(\mathcal{U}^-.L_{-2})_n - \dim \mathcal{U}^-_n = \mathrm{p}(n-1) + \mathrm{p}(n-2) - \mathrm{p}(n). \tag{27}$$

We define a linear map

$$\phi : \begin{cases} S & \longrightarrow & \mathcal{U}^- \times \mathcal{U}^- \\ \psi = \psi^1.L_{-1} = \psi^2.L_{-2} & \longmapsto & (\psi^1, \psi^2) \end{cases} \tag{28}$$

and then define $\tilde{L} := \phi(S)$ and $\tilde{L}_n := \phi(S_n)$. ϕ is well defined (Poincaré-Birkhoff-Witt for $\mathcal{U}^-$) and a vector space isomorphism, whence $\dim \tilde{L}_n = \dim S_n$. Now we define $\rho : \tilde{L}_n \to \mathrm{Hom}(V^1_{m-1} \times V^1_{m-2}, V^1_{m-n})$ by setting

$$\rho\left((\psi^1, \psi^2)\right)(v, w) := \psi^1.v - \psi^2.w.$$

Then $v^1 \in I$ is equivalent to

$$(v_1, v_2) \notin \ker \rho(\tilde{L}_{N_2})|_{V^1_{N_2-1} \times V^1_{N_2-2}},$$

where $\ker \rho(\tilde{L}_{N_2})|_{V^1_{N_2-1} \times V^1_{N_2-2}}$ is the intersection of the kernels of all its elements. We now proceed by proving

$$\dim \rho(\tilde{L}_{N_2})|_{V^1_{N_2-1} \times V^1_{N_2-2}} = \dim \tilde{L}_{N_2}.$$

To this end we assume the existence of $0 \neq (\chi, \psi) \in \tilde{L}_{N_2}$, such that $\chi.v - \psi.w = 0 \ \forall (v, w) \in V^1_{N_2-1} \times V^1_{N_2-2}$. With the involution (2) it follows that $\chi^\dagger.v^{(0)}, \psi^\dagger.v^{(0)} \in \mathrm{Rad}(\langle .,. \rangle)$. By theorem 2.9 $(\chi, \psi) = 0$, which contradicts the assumption.
We can now determine the dimension of the allowed parameter space $\tilde{\mathcal{V}}_{h^1,h^2} \subset V^1_{N_2-1} \times V^1_{N_2-2}$ for (v_1, v_2). With (27) and because of $\rho(\tilde{L}_{N_2})|_{V^1_{N_2-1} \times V^1_{N_2-2}} \subset \mathrm{Hom}(V^1_{N_2-1} \times V^1_{N_2-2}, \mathbb{C})$, it is

$$\dim \tilde{\mathcal{V}}_{h^1,h^2} = \dim V^1_{N_2-1} + \dim V^1_{N_2-2} - \dim \tilde{L}_{N_2} = \mathrm{p}(N_2). \tag{29}$$

After having determined the allowed parameter space we now examine which of these choices lead to isomorphic modules. Firstly, it is clear, that any two vermalike staggered modules whose data $\{v_0, v_1, v_2\}$ only differ by a nonzero scalar factor are isomorphic. We therefore fix the scaling of the upper LWV by demanding that $v_0 = v^2 = \chi_1.v^1$. Now two vermalike staggered modules M, M' are isomorphic if and only if there is a vector $\tilde{v} \in V^1_{N_2}$ so that $v_1' = v_1 + L_{-1}\tilde{v}$ and $v_2' = v_2 + L_{-2}\tilde{v}$. The space of all nonisomorphic vermalike staggered modules with lowest weights h^1 and h^2 is given by

$$\mathcal{V}_{h^1,h^2} = \tilde{\mathcal{V}}_{h^1,h^2} / (L_{-1}, L_{-2}).V^1_{N_2}. \tag{30}$$

By theorem 2.9 the dimension of $(L_{-1}, L_{-2}).V^1_{N_2}$ simply is

$$\dim V^1_{N_2} - 1 = \mathrm{p}(N_2) - 1. \tag{31}$$

Put together, this yields the resulting

Lemma 5.10 *With the notations of definition 4.12 the space of all nonisomorphic staggered modules with lowest weights h^1 and h^2 is a vector space $\mathcal{V}_{h^1,h^2}$ with dimension* $\dim \mathcal{V}_{h^1,h^2} = \mathrm{p}(N_2) - (\mathrm{p}(N_2) - 1) = 1.$ ✓

The above moduli space is most naturally parametrized in the following way: Obviously $\chi_1^\dagger.v^{(1)} = \alpha v^1$ with $\alpha \in \mathbb{C}$. $\alpha = 0$ is equivalent to M belonging to the equivalence class of $(v_0, v_1, v_2) = (\chi_1.v^1, 0, 0)$: Let $\{\psi_i.v^1,\ \psi_0 = \chi_1\}$ denote an orthogonal base of $V^1_{N_2}$ with respect to the Shapovalov form on V^1 with $\langle \psi_i.v^1, \psi_j.v^1 \rangle = s_i \delta_{ij}$. Theorem 2.9 implies that $s_i = 0 \Leftrightarrow i = 0$. If $\alpha = 0$ then $\tilde{v}^{(1)} := v^{(1)} - \sum_{k=1}^{\mathrm{p}(N_2)-1} s_k^{-1} \psi_k . {\psi_k}^\dagger v^{(1)}$ also is an upper lowest weight vector with $L_0.\tilde{v}^{(1)} = \chi_1.v^1$ and $L_{-1}.\tilde{v}^{(1)} = L_{-2}.\tilde{v}^{(1)} = 0$. On the other hand, due to $\chi_1.v^1$ belonging to the radical of $\langle .,. \rangle$, α does not depend on the choice of the representative (v_1, v_2).

We will now extend this result to the general case $h^{(0)} = h^1$ and $h^{(0)} = h^k, k \in \mathbb{N}^{\geq 2}$. One first observes

Lemma 5.11 *Let M be a vermalike staggered module with lowest weights h^1 and $h^k, k \in \mathbb{N}^{\geq 2}$. Then it is always possible to choose an upper LWV so that $v_1, v_2 \in V^{k-1} \subset M^{(0)}$.*

Proof: It suffices to show the following: If $v_1, v_2 \in V^j, j < k-1$, then there is a choice of upper lowest weight vector which yields $\tilde{v}_1, \tilde{v}_2 \in V^{j+1}$. Let $\{\psi_i\chi_j.v^j, \xi_l.v^j; 0 \leq i < \mathrm{p}(h^k - h^{j+1}) \leq l < \mathrm{p}(h^k - h^j)\}$ denote an orthogonal base of $V^j_{h^k-h^j}$ with respect to the Shapovalov form on V^j. As the Shapovalov form is nondegenerate on V^j/V^{j+1}, one has $\langle\psi_p\chi_j.v^j, \psi_q\chi_j.v^j\rangle = \langle\psi_p\chi_j.v^j, \xi_r.v^j\rangle = 0$ and $\langle\xi_r.v^j, \xi_s.v^j\rangle = s_r\delta_{rs},\ s_r \neq 0$. As $v_1, v_2 \in V^j$, we have $(\psi_i\chi_j)^\dagger.v^{(1)} = \chi_j^\dagger(\psi_i{}^\dagger.v^{(1)}) = 0$. Let $\tilde{v}^{(1)} := v^{(1)} - \sum_{r=\mathrm{p}(h^k-h^{j+1})}^{\mathrm{p}(h^k-h^j)-1} s_r^{-1}\xi_r.\xi_r{}^\dagger v^{(1)}$. Then by construction $\mathcal{U}^-_{h^k-h^j}.\tilde{v}^{(1)} = 0$ and therefore $\tilde{v}_1, \tilde{v}_2 \in \mathrm{Rad}(\langle.,.\rangle) = V^{j+1}$. ✓

We can now state our final result:

Theorem 5.12 *With the notations of definition 4.12 the space of all nonisomorphic staggered modules with lowest weights h^1 and $h^k, k \in \mathbb{N}^{\geq 2}$, is a vector space $\mathcal{V}_{h^1,h^k}$ with dimension* $\dim \mathcal{V}_{h^1,h^k} = 1$.

Proof: By lemmata 5.10 and 5.11 we have

$$\dim \mathcal{V}_{h^1,h^k} \leq 1. \tag{32}$$

Let $v_1, v_2 \in V^{k-1}$ be data of a staggered module with lowest weights h^{k-1}, h^k, which is not isomorphic to the staggered module with $v_1 = v_2 = 0$. The only thing which could prevent equality in (32) is the existence of a vector $\tilde{v} \in V^1 \backslash V^{k-1}$ so that $L_{-1}\tilde{v} = v_1$ and $L_{-2}\tilde{v} = v_2$. But then $\tilde{v} + V^{k+1}$ would be singular in V^1/V^{k-1}. By theorem 2.9 it follows that $\tilde{v} + V^{k+1} = 0 + V^{k+1}$ and therefore $\tilde{v} \in V^{k+1}$, which contradicts the assumption. ✓

Remark 5.13 *By using so-called central series [16], one can show the restrictions of corollary 5.3 even for the more general case of arbitrary extensions of a Verma module $V(h^{(0)}, c)$ by another Verma module $V(h^{(1)}, c)$. Almost the whole preceding argumentation goes through for the case of N-length 1, yielding two nonisomorphic modules – one nontrivial extension with N-length 1 and the (trivial) direct sum of the two modules:*
The vector space $\mathrm{Ext}^1(V(h^1,c), V(h^k,c))$ *of nonequivalent exact sequences*

$$0 \longrightarrow V(h^1,c) \xrightarrow{\beta} M \xrightarrow{\gamma} V(h^k,c) \longrightarrow 0,$$

where two sequences are equivalent if there is an $\mathcal{L}$-isomorphism $\alpha : M \to M'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(h^1,c) & \xrightarrow{\beta} & M & \xrightarrow{\gamma} & V(h^k,c) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \longrightarrow & V(h^1,c) & \xrightarrow{\beta'} & M' & \xrightarrow{\gamma'} & V(h^k,c) & \longrightarrow & 0 \end{array}$$

commutes, has complex dimension 2. The vector space $\mathcal{V}_{h^1,h^k}$ from theorem 5.12 is the one-dimensional affine subspace of $\mathrm{Ext}^1(V(h^1,c), V(h^k,c))$*, where $L_0^n v^{(1)}$ is fixed to a nonzero value. The linear subspace $\mathcal{V}$ parallel to $\mathcal{V}_{h^1,h^k}$ is exactly the subspace where* $\mathrm{N\text{-}length}(M) = 1$.
The space of all nonisomorphic modules M is then given by $\mathrm{P}(\mathrm{Ext}^1(V(h^1,c), V(h^k,c))) := \mathrm{Ext}^1(V(h^1,c), V(h^k,c))/\sim$*, where two sequences are equivalent ('$\sim$'), if there is a number $\delta \in \mathbb{C}^{\neq 0}$ and an $\mathcal{L}$-isomorphism $\alpha : M \to M'$, such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(h^1,c) & \xrightarrow{\beta} & M & \xrightarrow{\gamma} & V(h^k,c) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta\cdot id} & & \\ 0 & \longrightarrow & V(h^1,c) & \xrightarrow{\beta'} & M' & \xrightarrow{\gamma'} & V(h^k,c) & \longrightarrow & 0 \end{array}$$

commutes (this is a well defined equivalence relation on $\mathrm{Ext}^1(V(h^1,c),V(h^k,c))$*, since the old equivalence relation is just a special case* ($\delta = 1$) *of the new one)*†.
$\mathrm{P}(\mathrm{Ext}^1(V(h^1,c),V(h^k,c)))$ *splits as follows:*

$$\begin{aligned}\mathrm{P}(\mathrm{Ext}^1(V(h^1,c),V(h^k,c))) &= \mathcal{V}_{h^1,h^k}\,\dot\cup\,\mathrm{P}(\mathcal{V})\\ &= \mathcal{V}_{h^1,h^k}\,\dot\cup\,\{\text{extension with N-length }1\}\,\dot\cup\,\{V(h^1,c)\oplus V(h^k,c)\}.\end{aligned}$$

6 The maximal preserving submodules of JVMs

We are now prepared to explore the maximal preserving submodules of JVMs. As a first approach we prove the following theorem:

Theorem 6.1 *Let* $\tilde V(h^1,c)$ *a JVM and notations as in definition 4.12. For any* $n,m\in\mathbb{N}, m>n$ *there exists a submodule* $J\subset\tilde V(h^1,c)$ *with* $J^{(0)}=V^n$ *and* $J^{(1)}\cong V^m$.

Proof: Let $\tilde J := \mathcal{U}.\chi_{m-1}\dots\chi_1 v^{(1)}+\tilde V(h^1,c)^{(0)}$ denote the staggered submodule of $\tilde V(h^1,c)$ with lowest weights h^1 and h^m. According to lemma 5.11 $\tilde J$ possesses a staggered submodule $J\subset\tilde J\subset\tilde V(h^1,c)$ with lowest weights h^{m-1} and h^m. All other cases are given by the submodules $J+V^n$. ✓

By lemma 4.13 and the above theorem 6.1 a JVM V with maximal preserving submodule J falls into one of two classes: In any case $J^{(0)}=V^2$, but either $J^{(1)}\cong V^3$ or $J^{(1)}\cong V^2$, i.e. there exists an embedding $V(h^2,c)\hookrightarrow V(h^1,c)$. We now want to study the relationship between membership in one of the above classes and the lowest weight of the module. The means to do so are – again – provided by the Shapovalov form. In order to simplify notation we first define a projector $P:\mathcal{U}_0\to\mathcal{U}_0$ by linear continuation of the following settings:
For any monomial $u := L_{m_1}\dots L_{m_p}L_0^{k_0}C^{k_c}L_{n_1}\dots L_{n_q}\in\mathcal{U}_0$ with $m_1\ge\dots\ge m_p>0>n_1\ge\dots\ge n_q$ let

$$P(u)=\begin{cases}u & \text{if } p=q=0\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $u.v=P(u).v$ on a singular vector v.

Lemma 6.2 *With the notations of definition 4.12 the Kac determinant* $\det\langle .,.\rangle_{N_2}(h)$ *possesses a double zero at* $h=h^1$ *if and only if* $P(\chi_1^\dagger\chi_1)=\chi_1^*(L_0-h^1)^2$ *with* $\chi_1^*\in P(\mathcal{U}_0)$.

Proof: Let $\{\psi_0.v^1 := \chi_1.v^1, \psi_i.v^1; 1\le i<\mathrm{p}(N_2)\}$ be an orthogonal base of $V(h^1,c)$ with respect to $\langle .,.\rangle_{N_2}(h^1)$. We define the polynomials

$$p_{i,j}(h) := \langle \psi_i v_h, \psi_j v_h\rangle$$

where v_h is the lowest weight vector of $V(h,c)$. Then obviously

$$\det\langle .,.\rangle_{N_2}(h)=\det\begin{pmatrix} p_{0,0} & \cdots & p_{0,\mathrm{p}(N_2)-1}\\ \vdots & & \vdots\\ p_{\mathrm{p}(N_2)-1,0} & \cdots & p_{\mathrm{p}(N_2)-1,\mathrm{p}(N_2)-1}\end{pmatrix}. \tag{33}$$

As the first row and the first column vanish for $h=h^1$, the polynomials $p_{0,i}$ and $p_{i,0}$ have a common factor $(h-h^1)$. If we expand the determinant by the first column, we see that

$$\det\langle .,.\rangle_{N_2}(h)=p_{0,0}(h)p^*(h)+(h-h^1)^2\tilde p(h)$$

where $p^*(h^1)\neq 0$ because of theorem 2.9. Hence, the determinant possesses only a single zero at $h=h^1$ if and only if $p_{0,0}$ possesses only a single zero, which proves the assertion.✓

†For a given vector space V the space $\mathrm{P}(V)$ is the union of the corresponding projective space and one isolated point: $\mathrm{P}(V)=\mathbb{P}(V)\,\dot\cup\,\{0\}$.

By close examination of the Kac determinant formula (5), the knowledge, that each pair (r, s) with $h_{r,s} = h$ corresponds to a singular vector on level rs [13], and by careful study of the symmetries of $h_{r,s}$ one obtains the additional result:

Lemma 6.3 *As before, let $c = c_{1,q}$. Let $h_n := \frac{n^2-(q-1)^2}{4q}$. $\det \langle .,. \rangle_{N_2}(h)$ possesses a double zero at $h = h^1$ if and only if $h^1 = h_n$ where $n \neq 0$ is a multiple of q.*

Theorem 6.4 *With notations as in definition 4.12 let $\tilde{V}(h_n, c)$ be a JVM with lowest weight $h_n = \frac{n^2-(q-1)^2}{4q}$. The maximal preserving submodule $J \subset \tilde{V}(h_n, c)$ is a JVM if and only if $n \neq 0$ is a multiple of q.*

Proof: The maximal preserving submodule J is a JVM if and only if

$$\exists \tilde{v} \in V^1_{N_2} : \mathcal{U}^-.(\chi_1.v^{(1)} + \tilde{v}) = 0.$$

This is equivalent to

$$\chi_1^\dagger \chi_1.v^{(1)} = 0,$$

as due to $\chi_1.v^1$ belonging to the radical of the Shapovalov form

$$\nexists \tilde{v} \in V^1_{N_2} : \chi_1^\dagger \tilde{v} \neq 0.$$

If $\chi_1^\dagger \chi_1.v^{(1)} = 0$, a suitable choice is given by

$$\tilde{v} = -\sum_{i \neq 0} s_i^{-1} \psi_i \psi_i^\dagger \chi_1 v^{(1)},$$

where the ψ_i and s_i are defined as in the proof of lemma 6.2. With lemmata 6.2 and 6.3 the assertion follows. ✓

Remark 6.5 *The proof of theorem 6.4 uses special properties of the rank 2 case, namely the vanishing of $(L_0^n)^2$ on the upper lowest weight vector. Therefore, the possibility to embed a JVM into another, also is a genuine property of the rank 2 case.*

6.1 Characters

The character of an $\mathcal{L}$-module V is defined as

$$\chi_V(q) := \mathrm{tr}_V q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_h q^h \dim \mathrm{eigenspace}(L_0^d, h).$$

By lemma 2.6 the character of a Verma module therefore is given by

$$\chi_{V(h,c)} = q^{h-\frac{c}{24}} \sum_{k=0}^{\infty} \mathrm{p}(k) q^k = \frac{q^{\frac{1-c}{24}}}{\eta(q)} q^h,$$

where η is the Dedekind η-function $\eta(q) = q^{\frac{1}{24}} \prod_{n \in \mathbb{N}} (1 - q^n)$.

Corollary 6.6 *With the notations of definition 4.12 let $M(h^1, c)$ be the irreducible $\mathcal{L}$-module with lowest weight h^1. Its character is given by*

$$\chi_{M(h^1,c)} = \chi_{(V(h^1,c)/V(h^2,c))} = \frac{q^{\frac{1-c}{24}}}{\eta(q)} (q^{h^1} - q^{h^2}).$$

With the results of theorems 6.1 and 6.4 we obtain the

Corollary 6.7 *With the notations of definition 4.12 let $\tilde{M}(h,c)$ be a minimal JLWM. Its character is given by one of the following three formulas: If $\nexists r,s \in \mathbb{N}: \; h = h_{r,s}$, we have*

$$\chi_{\tilde{M}(h,c)} = 2\frac{q^{\frac{1-c}{24}}}{\eta(q)} q^h.$$

Else, $h = \frac{n^2-(q-1)^2}{4q}$ with $n \in \mathbb{N}^0$. If $n \neq 0$ and n is a multiple of q, then

$$\chi_{\tilde{M}(h,c)} = 2\frac{q^{\frac{1-c}{24}}}{\eta(q)} (q^{h^1} - q^{h^2}).$$

In all other cases,

$$\chi_{\tilde{M}(h,c)} = \frac{q^{\frac{1-c}{24}}}{\eta(q)} (2q^{h^1} - q^{h^2} - q^{h^3}).$$

Corollary 6.8 *The character of a vermalike staggered module V with lowest weights h^1 and h^k is given by*

$$\chi_V = \frac{q^{\frac{1-c}{24}}}{\eta(q)} (q^{h^1} + q^{h^k}).$$

If its characteristic parameter (see subsection 5.1) $\alpha = 0$, the character of the corresponding minimal staggered module $\tilde{V}$ is

$$\chi_{\tilde{V}} = \frac{q^{\frac{1-c}{24}}}{\eta(q)} (q^{h^1} + q^{h^k} - 2q^{h^{k+1}}).$$

In all other cases it is given by

$$\chi_{\tilde{V}} = \frac{q^{\frac{1-c}{24}}}{\eta(q)} (q^{h^1} + q^{h^k} - q^{h^{k+1}} - q^{h^{k+2}}).$$

7 Embeddings

The structure of minimal JVMs of rank 2 was completely resolved by theorems 6.1 and 6.4. This only involved the question, whether a JVM can be embedded into another as a *maximal* preserving submodule.

If, with the notations of definition 4.12 and theorem 6.4, $h = h_n$ with $n \neq 0$ and n a multiple of q, this question can easily be answered: As the lowest weight of the maximal preserving submodule of $\tilde{V}(h,c)$ again fulfills the above condition (see theorem 2.9), the complete embedding structure is given by

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & \vdots & & \\
 & & & & & & & & & & & & \downarrow & & \\
 & & & & & & & & J_{5,5} & \leftarrow & \cdots & \leftarrow & J_{5,\infty} & = & V^5 \\
 & & & & & & & & \downarrow & & & & \downarrow & & \\
 & & & & & & J_{4,4} & \leftarrow & J_{4,5} & \leftarrow & \cdots & \leftarrow & J_{4,\infty} & = & V^4 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & & & J_{3,3} & \leftarrow & J_{3,4} & \leftarrow & J_{3,5} & \leftarrow & \cdots & \leftarrow & J_{3,\infty} & = & V^3 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & J_{2,2} & \leftarrow & J_{2,3} & \leftarrow & J_{2,4} & \leftarrow & J_{2,5} & \leftarrow & \cdots & \leftarrow & J_{2,\infty} & = & V^2 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\tilde{V}(h,c) & \leftarrow & J_{1,2} & \leftarrow & J_{1,3} & \leftarrow & J_{1,4} & \leftarrow & J_{1,5} & \leftarrow & \cdots & \leftarrow & J_{1,\infty} & = & V^1,
\end{array}$$

where $J_{k,l}$ denotes a staggered submodule with lowest weights h^k and h^l. In general, one always has an embedding structure of the form

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & \vdots & & \\
 & & & & & & & & & & & & \downarrow & & \\
 & & & & & & & & (J_{5,5}) & \leftarrow & \cdots & \leftarrow & J_{5,\infty} & = & V^5 \\
 & & & & & & & & \downarrow & & & & \downarrow & & \\
 & & & & & & (J_{4,4}) & \leftarrow & J_{4,5} & \leftarrow & \cdots & \leftarrow & J_{4,\infty} & = & V^4 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & & & (J_{3,3}) & \leftarrow & J_{3,4} & \leftarrow & J_{3,5} & \leftarrow & \cdots & \leftarrow & J_{3,\infty} & = & V^3 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & (J_{2,2}) & \leftarrow & J_{2,3} & \leftarrow & J_{2,4} & \leftarrow & J_{2,5} & \leftarrow & \cdots & \leftarrow & J_{2,\infty} & = & V^2 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\tilde{V}(h,c) & \leftarrow & J_{1,2} & \leftarrow & J_{1,3} & \leftarrow & J_{1,4} & \leftarrow & J_{1,5} & \leftarrow & \cdots & \leftarrow & J_{1,\infty} & = & V^1,
\end{array}
$$

where the submodules in brackets may or may not exist. Unfortunately at the time being we cannot in general answer the question, which of these modules exist for given lowest weight h^1. Therefore we restrict ourselves to listing the restrictions we know:

(i) If $h^1 = h_n$ with $n \neq 0$ and n a multiple of q, all of the $J_{n,n}$ exist.

(ii) If $h^1 = h_0$, a submodule $J_{2,2}$ does not exist. If the submodule $J_{N,N}$ exists for any $N \in \mathbb{N}^{\geq 3}$, then all submodules $J_{n,n}$ with $n \geq N$ also exist.

(iii) If $h^1 = h_n$ with $n/q \notin \mathbb{N}^0$, a submodule $J_{2,2}$ does not exist. If, for any $n \in \mathbb{N}$, the submodule $J_{n,n}$ exists, then no submodule $J_{n+1,n+1}$ can exist.

8 Rational models

It is a natural question, whether there are rational conformal field theories at values of the central charge from the logarithmic series. Of course, the notion of rationality here has to be broadened to include indecomposable representations of the underlying $\mathcal{W}$-algebra (for a short and exact definition of $\mathcal{W}$-algebra see e.g. [10]). In the mathematical literature one often defines a rational theory to be decomposable into finitely many *irreducible* representations, which close under fusion – this clearly cannot be the case for the above central charges. We will later see, in which way the usual definition of rationality has to be broadened to include logarithmic models (definition 8.14).
While it is not possible to find rational theories with respect to the Virasoro algebra, at least for some of the central charges in the logarithmic series one indeed finds models of larger $\mathcal{W}$-algebras, which are rational in this slightly broadened sense.
The right candidates for such rational theories were already identified by M. Flohr [21, 22]. He found that the characters of the known representations of a suitably chosen $\mathcal{W}$-algebra span finite dimensional representations of the modular group $SL(2,\mathbb{Z})$. The problem with these representations was that they necessarily include "characters" with logarithmic terms in q, which, at least with the usual definition of a character, cannot occur. In fact, one can find finite dimensional representations of the modular group without the inclusion of logarithmic "characters" if one does not start with the set of usual lowest weight representations of the algebra but rather with some suitable extensions thereof.
To see this, we will now study the simplest example of such a rational logarithmic model. It is based on the triplet algebra $\mathcal{W}(2,3^3)$ at $c=-2$. This algebra was found by H. Kausch [32] and

is spanned by the modes of the Virasoro field and three additional primary fields of conformal weight three. The algebra is given by the commutation relations

$$
\begin{aligned}
[L_m, L_n] &= (n-m)L_{m+n} - \frac{1}{6}(n^3-n)\delta_{n+m,0}, && (34a)\\
[L_m, W^a_n] &= (n-2m)W^a_{m+n}, && (34b)\\
\left[W^a_m, W^b_n\right] &= g^{ab}\,(4\,p_{334}(m,n)\Lambda_{m+n} + 3\,p_{335}(m,n)L_{m+n} \\
&\quad - \left(\begin{array}{c} n+2 \\ 5 \end{array}\right)\delta_{m+n,0}\Big) \\
&\quad + f^{ab}_c\left(5\,p_{333}(m,n)W^c_{m+n} + \frac{12}{5}\Omega^c_{m+n}\right) && (34c)
\end{aligned}
$$

where $a,b\in\{1,2,3\}$, $\Lambda := \mathcal{N}(L,L) = N(L,L) - 3/10\,\partial^2 L$ and $\Omega^a := \mathcal{N}(W^a,L) = N(W^a,L) - 3/14\,\partial^2 W^a$ are quasiprimary normal ordered fields (for notational conventions see the appendix). The p_{ijk} are universal polynomials:
$p_{332}(m,n) = \frac{n-m}{60}(2m^2+2n^2-mn-8)$, $p_{333}(m,n) = \frac{1}{14}(2m^2+2n^2-3mn-4)$ and $p_{334}(m,n) = \frac{n-m}{2}$. f^{ab}_c and g^{ab} are the structure constants and standard symmetric bilinear form of $su(2)$ (the latter is half the Killing form on $su(2)$, i.e. in the standard base with $f^{ab}_c = i\varepsilon_{abc}$ one has $g^{ab} = \delta_{a,b}$).
Before we study the above-mentioned rational model, we must first slightly generalize our definitions from sections 2 and 3. For the sake of notational simplicity we will concentrate on the above defined $\mathcal{W}$-algebra $\mathcal{W}(2,3^3)$ at $c=-2$. The generalizations to other $\mathcal{W}$-algebras will be obvious.
We first remark that irreducible modules of a given $\mathcal{W}$-algebra are not necessarily lowest weight modules. The role of lowest weight modules will be played by a slightly broader class of modules, which are based on irreducible representations of the subalgebra of zero modes (in the case of $\mathcal{W}(2,3^3)$ the subalgebra generated by $\{L_0, W^1_0, W^2_0, W^3_0\}$).

Definition 8.1 *Let* $\mathrm{Mod}_{\mathcal{W}}$ *be the category of* $\mathcal{W}(2,3^3)$*-modules, which as* $\mathcal{L}$*-modules belong to* $\mathrm{Mod}_{\mathcal{L}}$*. Its objects will from now on simply be called* $\mathcal{W}(2,3^3)$*-modules.*

Lemma 8.2 *One has* $[L^n_0, W^a_k] = 0$*. Furthermore*

$$W^a_k : \mathrm{eigenspace}(L^d_0, h) \to \mathrm{eigenspace}(L^d_0, h+k). \tag{35}$$

For an indecomposable $\mathcal{W}(2,3^3)$*-module* V *one therefore has*

$$V = \bigoplus_{n=0}^{\infty} \mathrm{eigenspace}(L^d_0, h_{min}+n). \tag{36}$$

Proof: Analogous to subsection 3.1. ✓

Definition 8.3 *A* $\mathcal{W}(2,3^3)$*-module* $M \in \mathrm{Mod}_{\mathcal{W}}$ *is called* ***generalized lowest weight module (GLWM)*** *if there is a linear subspace* $M_0 \subset M$ *such that*

(i) $\forall n\in\mathbb{N}, v\in M_0, a\in\{1,2,3\} : L_{-n}v = W^a_{-n}v = 0$*,*

(ii) M_0 *is an irreducible* $\mathcal{W}_0$*-module,*

(iii) $M = \mathcal{U}_{\mathcal{W}}.M_0$*,*

where $\mathcal{U}_\mathcal{W}$ denotes the universal enveloping algebra of $\mathcal{W}(2,3^3)$. If $\dim M_0 = 1$, M is called ***lowest weight module*** *or* ***singlet module****. For $\dim M_0 = 2$, M is called* ***doublet module****.*

Lemma and Definition 8.4 *Let $M \in \mathrm{Mod}_\mathcal{W}$ be a generalized lowest weight module. Then $\exists h \in \mathbb{C}$ such that $\forall v \in M_0 : L_0.v = hv$. h is called* ***lowest weight*** *of the module and the elements of a base of the* ***lowest weight space*** *M_0 are called* ***lowest weight vectors****.*

Proof: From (34) we have $[L_0, W^a_0] = 0$ (this is true for arbitrary primary fields W^a). With Schur's lemma the assertion follows. ✓

Lemma 8.5 *Let $M \in \mathrm{Mod}_\mathcal{W}$ be indecomposable and $L_0 M_0 = hM_0, h \in \mathbb{C}$. Then the $\mathcal{W}_0$-module M_0 is not only indecomposable, but irreducible.*

Proof: Because of $W^a_{-n}M_0 = L_{-n}M_0 = 0$, $n \in \mathbb{N}$, for $v \in M_0$ one has $\mathcal{U}_\mathcal{W} v \cap M_0 = \mathcal{U}(\mathcal{W}_0)v$. Hence, the $\mathcal{W}_0$-module M_0 is indecomposable.

For $v \in M_0$ one easily calculates

$$\Omega^a_0 v = \left(h + \frac{3}{7}\right) W^a_0 v.$$

With (34) it is clear, that the representation of $\langle W^1_0, W^2_0, W^3_0 \rangle$ on M_0 is just a representation of $su(2)$. $su(2)$ being semisimple, its finite dimensional representations are completely reducible by Weyl's theorem (see e.g. [30]). Therefore, M_0 is both indecomposable and completely reducible, hence irreducible. ✓

Lemma 8.6 *Let $V \in \mathrm{Mod}_\mathcal{W}$ be a $\mathcal{W}(2,3^3)$-module. Then there exists a submodule $M \subset V$ which is a generalized lowest weight module.*

Proof: Without loss of generality suppose V to be indecomposable. Analogously to the proof of lemma 3.5, it possesses a submodule M with $L_0 M_0 = hM_0, h \in \mathbb{C}$. M can also be chosen to be indecomposable. With lemma 8.5 M_0 then is an irreducible $\mathcal{W}_0$-module and $\mathcal{U}_\mathcal{W} M_0 \subset V$ is a GLWM. ✓

Using the above lemma, we may define the **lowest weight length** of a $\mathcal{W}(2,3^3)$-module analogously to definition 3.7, if we substitute *lowest weight module* by *generalized lowest weight module*. The **length** and **nilpotency length** of a $\mathcal{W}(2,3^3)$-module are defined analogously to definitions 3.4 and 3.7.

As the analogues to staggered modules and JLWMs in the pure Virasoro case we define

Definition 8.7 *An indecomposable $\mathcal{W}(2,3^3)$-module $M \in \mathrm{Mod}_\mathcal{W}$ is called* ***staggered module****, if* N-length(M) = LW-length$(M) = N \in \mathbb{N}^{\geq 2}$*. The number N is called its* ***rank****.*

Definition 8.8 *A staggered $\mathcal{W}(2,3^3)$-module M of rank N is called* ***Jordan lowest weight module*** *if $M = \mathcal{U}_\mathcal{W}.M_0$ and $\forall v \in M_0 : L_0 v = hv$.*

Definition 8.9 *A staggered $\mathcal{W}(2,3^3)$-module M of rank 2 is called* ***strictly staggered****, if it is not a JLWM.*

As we will see later, it also becomes necessary to introduce yet another class of modules:

Definition 8.10 *An indecomposable $\mathcal{W}(2,3^3)$-module M with* N-length$(M) = 2$ *is called* ***generalized staggered module*** *if $M^{(1)}$ is a GLWM.*

8.1 Null field relations

On first sight the above definitions seem to admit a much larger class of modules than in the Virasoro case. This is in fact not true, as modules of a given $\mathcal{W}$-algebra must meet some restrictions which do not occur in the pure Virasoro case.
In general the algebra is only consistent (fulfills the Jacobi identities) due to certain null states in the vacuum representation. The existence of these null states, corresponding to so-called *null fields*, i.e. fields with vanishing two-point functions, poses additional restrictions on representations of the algebra, namely the vanishing of these null fields [17]. The algebra $\mathcal{W}(2,3^3)$ is only associative due to the following null states in the vacuum representation:

$$\begin{aligned} A^a &= \left(2L_3W_3^a - \frac{4}{3}L_2W_4^a + W_6^a\right)|0\rangle \qquad (37a) \\ B^{ab} &= W_3^bW_3^a|0\rangle - g^{ab}\left(\frac{8}{9}L_2^3 + \frac{19}{36}L_3^2 + \frac{14}{9}L_4L_2 - \frac{16}{9}L_6\right)|0\rangle \\ &\quad -f_c^{ab}\left(-2L_2W_4^c + \frac{5}{4}W_6^c\right)|0\rangle. \qquad (37b) \end{aligned}$$

Definition 8.11 *Any $\mathcal{W}(2,3^3)$-module M fulfills*

$$\forall v \in M, k \in \mathbb{Z} : \mathcal{A}_k^a v = \mathcal{B}_k^{ab} v = 0, \tag{38}$$

*where $\mathcal{A}^a$ and $\mathcal{B}^{ab}$ denote the fields corresponding to the null vectors A^a and B^{ab}, respectively (see the appendix for details). The property (38) is called **admissibility**, the module **admissible**.*

8.2 Generalized lowest weight modules

We now want to study, which admissible generalized lowest weight modules can exist. Admissible modules must be annihilated by the null modes of $\mathcal{A}^a$ and $\mathcal{B}^{ab}$, respectively – in particular their lowest weight spaces must be annihilated, where the action of the zero modes is especially easy to calculate. For $v \in M_0$ one obtains

$$\mathcal{B}^{ab}v = \left(W_0^aW_0^b - g^{ab}\frac{1}{9}L_0^2(8L_0+1) - f_c^{ab}\frac{1}{5}(6L_0-1)W_0^c\right)v = 0. \tag{39}$$

The relation $\mathcal{A}^a v = 0$ is satisfied identically. Further restrictions are obtained if one examines higher modes of the null fields. The study of the equations $W_{-1}^a\mathcal{B}_1^{bc}v$, together with equation (39), after some lengthy but straightforward algebra yields the result

$$W_0^a(8L_0-3)(L_0-1)v = 0,$$

which after multiplication by W_0^a together with (39) forces

$$L_0^2(8L_0+1)(8L_0-3)(L_0-1)v = 0. \tag{40}$$

This restricts the lowest weights of generalized lowest weight modules to

$$h \in \left\{-\frac{1}{8}, 0, \frac{3}{8}, 1\right\}.$$

We now have to determine, which irreducible representations of the zero mode algebra correspond to these values of h: By redefining

$$\tilde{W}_0^a := \frac{5}{2(6h-1)}W_0^a, \tag{41}$$

one obtains on M_0 the $su(2)$-algebra

$$[\tilde{W}_0^a, \tilde{W}_0^b] = f_c^{ab} \tilde{W}_0^c.$$

Its Casimir operator $\sum_{a,b} g_{ab} \tilde{W}_0^a \tilde{W}_0^b$ can then easily be evaluated using (39):

h	$-\frac{1}{8}$	0	$\frac{3}{8}$	1
$\sum_{a,b} g_{ab} \tilde{W}_0^a \tilde{W}_0^b$	0	0	$\frac{3}{4}$	$\frac{3}{4}$

table 8.2.1: Admissible GLWMs

From this result we conclude, that there exist at most four inequivalent admissible irreducible $\mathcal{W}(2,3^3)$-modules[‡]: Two singlet modules at $h = -\frac{1}{8}, 0$ (from now on called $V_{-1/8}^{\mathcal{W}}$ and $V_0^{\mathcal{W}}$) and two doublet modules at $h = \frac{3}{8}, 1$ ($V_{3/8}^{\mathcal{W}}$ and $V_1^{\mathcal{W}}$). The singlet module at $h = 0$ is of course just the vacuum representation. All four modules have been obtained in [33] using a free field construction.

The question, which *reducible* admissible generalized lowest weight modules might exist, is also easily answered, as their maximal proper submodules must again be admissible. In particular, the lowest weight space of a maximal proper submodule must fulfill equation (40), which only allows a reducible generalized LWM at $h = 0$. In fact, such a module does exist: The $\mathcal{W}(2,3^3)$-Verma module $\mathcal{U}_{\mathcal{W}} / \langle L_i, W_i^a; i \leq 0, a = 1,2,3 \rangle$ with lowest weight 0 possesses two generalized lowest weight submodules with lowest weight 1 (doublet modules). It cannot possess any non-trivial submodules with trivial intersection with the first two levels, since such a submodule cannot be admissible. We denote this module by $\tilde{V}_0^{\mathcal{W}}$.

For future convenience we will now fix a choice of base for $su(2)$ to a Cartan-Weyl base $\{l^+, l^-, l^0\}$, such that the nonvanishing structure constants are given by $f_0^{+-} = -f_0^{-+} = 2$, $f_\pm^{0\pm} = -f_\pm^{\pm 0} = \pm 1$ and the nonvanishing coefficients of the standard bilinear form become $g^{00} = 1$, $g^{\pm\mp} = 2$. The representation matrices in the two-dimensional irreducible representation of $su(2)$ then, with a suitably chosen base $\{v^+, v^-\}$, are given by

$$\tau^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \tau^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; \tau^0 := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With v the lowest weight vector of the above reducible GLWM, a choice of lowest weight vectors for the two $V_1^{\mathcal{W}}$ submodules is then given by

$$v_1^+ := \psi_1^+ v, \qquad \psi_1^+ := W_1^+ \tag{42a}$$

$$v_1^- := \psi_1^- v, \qquad \psi_1^- := \left(\frac{1}{2} L_1 - W_1^0 \right) \tag{42b}$$

and

$$v_2^+ := \psi_2^+ v, \qquad \psi_2^+ := \left(\frac{1}{2} L_1 + W_1^0 \right) \tag{43a}$$

$$v_2^- := \psi_2^- v, \qquad \psi_2^- := W_1^-. \tag{43b}$$

The above choice of base of course is somewhat arbitrary, but will prove useful later. In agreement with (41) one calculates

$$W_0^a v_i^b = 2 \tau^a v_i^b, \; L_0 v_i^b = v_i^b. \tag{44}$$

[‡]Uniqueness is proven analogously to theorem 2.6/corollary 2.7.

8.3 Jordan lowest weight modules

Admissibility poses even stronger restrictions on JLWMs than on GLWMs. For the derivation of equation (40) we only used, that L_{-n} and W^a_{-n} annihilate the lowest weight space. Therefore, (40) is also valid on the lowest weight space of an admissible JLWM. Hence, the only admissible JLWMs of $\mathcal{W}(2,3^3)$ have lowest weight 0 and nilpotency length 2.
One further calculates, that there is no JLWM with either both stages isomorphic to $V_0^{\mathcal{W}}$ or both stages isomorphic to $\tilde{V}_0^{\mathcal{W}}$ (calculations up to level 2 are sufficient to prove this). Therefore, there exists *only one* $\mathcal{W}(2,3^3)$-JLWM (denoted by $V_0^{\mathcal{W}*}$) and $(V_0^{\mathcal{W}*})^{(0)} = V_0^{\mathcal{W}}$, $(V_0^{\mathcal{W}*})^{(1)} = \tilde{V}_0^{\mathcal{W}}$.

8.4 Staggered modules

Equation (40) only allows strictly staggered modules with lowest weights 0 and 1. Now assume the existence of such a staggered module and let $v^{(0)}$, $v^{(1)+}$ and $v^{(1)-}$ its lower and upper LWVs. Obviously, $L_0^n v^{(1)\pm}$ and $(W_0^a - 2\tau^a)v^{(1)\pm}$ must be singular (all vectors on the first level of the lower module are) and therefore

$$\begin{aligned} 0 &= W_0^a L_{-1} v^{(1)\pm} = [W_0^a, L_{-1}]v^{(1)\pm} + L_{-1}W_0^a v^{(1)\pm} \\ &= -2W_{-1}^a v^{(1)\pm} + L_{-1}W_0^a v^{(1)\pm} = -2W_{-1}^a v^{(1)\pm} + 2L_{-1}\tau^a v^{(1)\pm} \end{aligned} \tag{45}$$

$$\Longrightarrow \quad W_{-1}^a v^{(1)\pm} = \frac{1}{2} L_{-1} W_0^a v^{(1)\pm} = L_{-1}\tau^a v^{(1)\pm}. \tag{46}$$

Hence, the action of L_{-1} on $v^{(1)\pm}$ completely determines the action of W^a_{-1} on $v^{(1)\pm}$.
We now must check, whether such a module can be admissible. To this end we must study the restrictions coming from the null fields $\mathcal{A}^a$, $\mathcal{B}^{ab}$, where we now must take into account all field modes W_n^a, L_n with $n \geq -1$. If one evaluates the constraint from $\mathcal{A}_0^a v^{(1)\pm}$, one finds that

$$W_1^a L_{-1} v^{(1)\pm} = L_1 W_{-1}^a v^{(1)\pm}. \tag{47}$$

Therefore, either $L_{-1}v^{(1)\pm} = W_{-1}^a v^{(1)\pm} = 0$ or $L_1 v^{(0)} = \lambda W_1^a v^{(0)}, \lambda \in \mathbb{C}$. The first case is not allowed, since it would mean the existence of an admissible JLWM with lowest weight 1 (c.f. section 8.3). The second implies, since there is no *singlet* module at $h = 1$, that

$$L_1 v^{(0)} = W_1^a v^{(0)} = 0. \tag{48}$$

Hence, $L_0^n v^{(1)\pm} = 0$, which is not allowed for staggered modules.

8.5 Generalized staggered modules

While a strictly staggered $\mathcal{W}(2,3^3)$-module does not exist, we may still expect to find generalized staggered modules as defined in definition 8.10. Assume the existence of such a module M, which is not a staggered module. As before, equation (40) forces $M^{(1)} = V_1^{\mathcal{W}}$. We further assume M to be minimal in the sense, that it does not contain any proper submodule, that also is a generalized staggered module.
Right from the beginning we know, that in the decomposition of the lower module $M^{(0)}$ into irreducible modules only the modules $V_0^{\mathcal{W}}$ and $V_1^{\mathcal{W}}$ can occur. In addition we know, that in the above decomposition the module $V_0^{\mathcal{W}}$ *must* occur, because there is no admissible JLWM with lowest weight 1. Therefore we know that $L_0 M_0 = 0$. From subsection 8.2 we further know, that $W_0^a M_0 = 0$. We will now successively study the further restrictions on the lower module posed by admissibility.
Let us choose representatives $v^{(1)+}, v^{(1)-} \in M_1$ of the lowest weight vectors of $M^{(1)}$. Let $\tilde{M}^{(0)} := \mathcal{U}_{\mathcal{W}}\langle L_{-1}v^{(1)\pm}, W_{-1}^a v^{(1)\pm}\rangle$. Obviously, $\tilde{M}^{(0)}$ is a submodule of $M^{(0)}$. In fact, $\tilde{M}^{(0)} = M^{(0)}$: Otherwise $M/\tilde{M}^{(0)}$ would either contain an admissible JLWM with lowest weight 1 as a submodule, or it would do so after modding out $\mathcal{U}_{\mathcal{W}}$.eigenspace$(L_0, 0)$.

With equation (46) we find, that $\dim M_0^{(0)}$ can be at most 2. As there are no strictly staggered modules of $\mathcal{W}(2,3^3)$, we have $\dim M_0^{(0)} = 2$. A base of $M_0^{(0)}$ is given by the two vectors

$$v^{(0)\pm} := L_{-1}v^{(1)\pm}. \tag{49}$$

Equation (47) then implies that $\dim M_1^{(0)} = 2$ and a base thereof is given by

$$\tilde{v}^{(0)\pm} := L_1 v^{(0)\pm} = L_1 L_{-1} v^{(1)\pm}. \tag{50}$$

More precisely, using equation (47) and the commutator relations (34), one finds

$$W_1^a L_{-1} v^{(1)\pm} = L_1 W_{-1}^a v^{(1)\pm} = -\frac{1}{2} L_1 L_{-1} W_0^a v^{(1)\pm}, \tag{51}$$

and therefore

$$\begin{aligned}
\psi_1^{\pm} v^{(0)+} &= 0, && (52a)\\
\psi_2^{\pm} v^{(0)-} &= 0, && (52b)\\
\psi_2^{\pm} v^{(0)+} &= \psi_1^{\pm} v^{(0)-} = \tilde{v}^{(0)\pm}. && (52c)
\end{aligned}$$

The lower module therefore is isomorphic to

$$\left(\tilde{V}_0^{\mathcal{W}} \oplus \tilde{V}_0^{\mathcal{W}}\right)/A,$$

where

$$A := \mathcal{U}_{\mathcal{W}}\psi_1^{\pm}(v,0) + \mathcal{U}_{\mathcal{W}}\psi_2^{\pm}(0,v) + \mathcal{U}_{\mathcal{W}}(\psi_2^{\pm}v, -\psi_1^{\pm}v)$$

and v is a LWV of $\tilde{V}_0^{\mathcal{W}}$.

Using $\mathcal{B}_0^{ab}v^{(1)\pm} = 0$, $W_{-1}^a\mathcal{B}^{cd}v^{(1)\pm} = 0$, $(L_0^n)^2 = 0$ and $L_0^n M^{(0)} = 0$, after some rather lengthy algebra one finds

$$14 W_1^a W_{-1}^b v^{(1)\pm} - 9 f_c^{ab} W_1^c L_{-1} v^{(1)\pm} - f_c^{ab} L_0^n W_0^c v^{(1)\pm} = 0.$$

With (46) and (51) this implies

$$\begin{aligned}
0 &= L_0^n W_0^c v^{(1)\pm} + L_1 L_{-1} W_0^c v^{(1)\pm}\\
&\Longrightarrow L_0^n v^{(1)\pm} = -\tilde{v}^{(0)\pm}. && (53)
\end{aligned}$$

In order to fix the structure of M completely, we now only have to determine the action of W_0^a on $v^{(1)\pm}$. With (34) and (47) one computes

$$\begin{aligned}
[W_0^a W_0^b] v^{(1)\pm} &= 2 f_c^{ab}\left(-\frac{1}{5}W_0^c + \frac{12}{5}W_1^c L_{-1} + \frac{6}{5}L_0 W_0^c\right) v^{(1)\pm}\\
&\overset{(51,53)}{=} 2 f_c^{ab} W_0^c v^{(1)\pm}\\
&\Longrightarrow \forall v \in M_1 : [W_0^a W_0^b] v = 2 f_c^{ab} W_0^c v. && (54)
\end{aligned}$$

This means, that $v^{(1)\pm}$ can be *chosen* (one always has the freedom of choice $v^{(1)\pm} \mapsto v^{(1)\pm} + v, v \in M_1^{(0)}$) in such a way that

$$W_0^a v^{(1)\pm} = 2\tau^a v^{(1)\pm}.$$

This module, which was also found by M.R. Gaberdiel and H.G. Kausch in the fusion of lowest weight $\mathcal{W}(2,3^3)$-modules [28], will be denoted by $V_1^{\mathcal{W}*}$. Note, that admissibility fixes the structure of the module completely and (in contrast to the pure Virasoro case) the moduli space of generalized staggered modules therefore consists of one point only.

We now examine the possibilities to extend a GLWM with lowest weight 0 by another GLWM with lowest weight 1 yielding an indecomposable module of nilpotency length 1. With equations (48) and (46) we conclude, that the general module of this form is given by

$$V_1^{\mathcal{W}*}/\mathcal{U}_{\mathcal{W}}.\left(\lambda v^{(0)+}+\mu v^{(0)-}\right),\ (\lambda,\mu)\in\mathbb{C}^2\backslash\{(0,0)\}. \tag{55}$$

Two of these modules are isomorphic if and only if $\lambda_1\mu_2=\lambda_2\mu_1$. Thus the moduli space of inequivalent modules of this form is given by the Riemannian sphere $\mathbb{C}\cup\{\infty\}$.

8.6 Other modules

Of course it is always possible to construct new modules from old ones by taking direct sums and modding out submodules (as an example, the module $(V_1^{\mathcal{W}*})^{(0)}$ was constructed in such a way from two copies of $\tilde{V}_0^{\mathcal{W}}$). This naturally produces an infinite variety of $\mathcal{W}(2,3^3)$-modules. The remaining question is, whether there are $\mathcal{W}(2,3^3)$-modules that are neither exhibited in the preceding subsections nor can be constructed from such using the aforementioned operations.

Definition 8.12 *Let $\mathcal{M}$ be the moduli space of all inequivalent $\mathcal{W}(2,3^3)$-modules together with the three operations 'taking submodules' (Sub), 'taking direct sums' (Sum) and 'modding out submodules' (Mod).*

Theorem 8.13 *The moduli space $\mathcal{M}$ is generated from the set*

$$\left\{V_{-1/8}^{\mathcal{W}},V_0^{\mathcal{W}*},V_{3/8}^{\mathcal{W}},V_1^{\mathcal{W}*}\right\} \tag{56}$$

by the three operations (Sub), (Sum) and (Mod).

Proof: Let M be a $\mathcal{W}(2,3^3)$-module. Without loss of generality we can assume M to be indecomposable. First assume that one of the irreducible modules M^k from the filtration (17) is either isomorphic to $V_{-1/8}^{\mathcal{W}}$ or to $V_{3/8}^{\mathcal{W}}$. As the levels of an indecomposable module are integer spaced, all of the M^k are isomorphic to $V_{-1/8}^{\mathcal{W}}$ or $V_{3/8}^{\mathcal{W}}$, respectively. Since there are no JLWMs with lowest weight $-1/8$ and $3/8$, M is isomorphic to $V_{-1/8}^{\mathcal{W}}$ or $V_{3/8}^{\mathcal{W}}$ and we are through.

Now assume, that M is an admissible indecomposable $\mathcal{W}(2,3^3)$-module, which is not generated from the above set of modules (56). We will prove this to be impossible in three steps:

(a) We first investigate the case N-length$(M)=1$. If M is not obtained by one of the above operations, it must be an extension of some combination of lowest weight modules with lowest weight 0 by one ore more lowest weight module(s) with lowest weight 1. Without loss of generality we can assume M to be generated by the representatives v^+,v^- of the lowest weight vectors of *one* GLWM with lowest weight 1. Then equations (47) and (53) imply that $\mathcal{U}_{\mathcal{W}}L_{-1}v^\pm+\mathcal{U}_{\mathcal{W}}W_{-1}^a v^\pm$ is either isomorphic to $V_0^{\mathcal{W}}$ or isomorphic to $V_0^{\mathcal{W}}\oplus V_0^{\mathcal{W}}$. M can therefore be constructed from one or more copies of $V_1^{\mathcal{W}*}/V_1^{\mathcal{W}}$.

(b) We now turn to the case N-length$(M)=2$. Without loss of generality we may assume that the upper module of M is indecomposable. If $M^{(1)}$ is either a GLWM with lowest weight 0 or one with lowest weight 1, we are through by subsections 8.3, 8.4 and 8.5. Thus assume $M^{(0)}$ to be an extension of some combination of GLWMs

with lowest weight 0 by $V_1^{\mathcal{W}}$. We will now show that this is impossible. Let $w^\pm$ be representatives of the lowest weight vectors of $V_1^{\mathcal{W}}$. We first remark, that $L_0^n w^\pm \neq 0$, because otherwise $L_0^n W_{-1}^a w^\pm = W_{-1}^a L_0^n w^\pm = 0$, $L_0^n L_{-1} w^\pm = L_{-1} L_0^n w^\pm = 0$ and thus N-length$(M) = 1$. If $L_0^n W_{-1}^a w^\pm = L_0^n L_{-1} w^\pm = 0$, M contains a JLWM with lowest weight 1, which is not admissible. We conclude, that M contains a JLWM with lowest weight 0 as a submodule. Let v be a representative of the lowest weight vector of the upper weight 0 - GLWM. The extension by $V_1^{\mathcal{W}}$ forces

$$L_1 v = W_1^a v = 0 \tag{57}$$

(c.f. equation (48)). On the other hand, v is upper LWV of the above-mentioned JLWM, which forces $\mathcal{U}_{\mathcal{W}}.v/\mathcal{U}_{\mathcal{W}}.L_0 v = \tilde{V}_0^{\mathcal{W}}$ (note the difference to $(\mathcal{U}_{\mathcal{W}}.v)^{(1)} = V_0^{\mathcal{W}}$) in contradiction to (57).

(c) Last but not least we have to deal with the case N-length$(M) > 2$. We will show, that such a module cannot exist. To prove this it suffices to examine the case N-length$(M) = 3$, because any module with higher nilpotency length would have to contain a submodule with nilpotency length 3. By subsection 8.3, such a module cannot be a JLWM. First note, that the submodule $L_0^n.M \subset M$ must contain one or more linearly independent vectors $v_i, i \in I$, with $L_0^d v_i = 0$, $L_0^n v_i \neq 0$, since otherwise the module $M/M^{(0)}$ cannot be admissible. Hence, the $\mathcal{W}(2,3^3)$-module $\tilde{M} := M/\mathcal{U}_{\mathcal{W}}.\langle L_0.v_i, i \in I\rangle$ must have nilpotency length 2, because $L_0^n.M/\mathcal{U}_{\mathcal{W}}.\langle L_0^n.v_i, i \in I\rangle$ has nilpotency length 1 and $\mathcal{U}_{\mathcal{W}}.\langle v_i, i \in I\rangle \subset M^{(0)}$. On the other hand, in $\tilde{M}$ we have $\forall i \in I : \mathcal{U}_{\mathcal{W}} v_i \equiv \tilde{V}_0^{\mathcal{W}}$, which is impossible for the lower module of an indecomposable nilpotency 2 module.

✓

Motivated by this result we slightly extend the usual definition of 'rational model' as follows:

Definition 8.14 *A* ***rational model*** *of a $\mathcal{W}$-algebra or* ***rational $\mathcal{W}$-algebra*** *is a $\mathcal{W}$-algebra $\mathcal{A}$ which fulfills the following condition: There exists a finite set of $\mathcal{A}$-modules, from which all other $\mathcal{A}$-modules can be obtained by taking submodules, factor modules and direct sums.*

8.7 Characters and modular properties

The character of a $\mathcal{W}(2,3^3)$-module is defined to be its character as an $\mathcal{L}$-module. The characters of the irreducible $\mathcal{W}(2,3^3)$-modules have been known for quite a while [21, 22, 33] and are given by

$$\chi_{V_0^{\mathcal{W}}}(q) = \sum_{k=0}^{\infty}(2k+1)\chi_{M(h_{4k+3,1},-2)}(q) = \frac{1}{2\eta(q)}(\Theta_{1,2}(q) + (\partial\Theta)_{1,2}(q)), \tag{58a}$$

$$\chi_{V_{-1/8}^{\mathcal{W}}}(q) = \sum_{k=0}^{\infty}(2k+1)\chi_{M(h_{4k+2,1},-2)}(q) = \frac{1}{\eta(q)}\Theta_{0,2}(q), \tag{58b}$$

$$\chi_{V_{3/8}^{\mathcal{W}}}(q) = \sum_{k=0}^{\infty}(2k+2)\chi_{M(h_{4k+4,1},-2)}(q) = \frac{1}{\eta(q)}\Theta_{2,2}(q), \tag{58c}$$

$$\chi_{V_1^{\mathcal{W}}}(q) = \sum_{k=0}^{\infty}(2k+2)\chi_{M(h_{4k+5,1},-2)}(q) = \frac{1}{2\eta(q)}(\Theta_{1,2}(q) - (\partial\Theta)_{1,2}(q)), \tag{58d}$$

where the $\Theta_{\lambda,k}$ are the Jacobi-Riemannian theta functions

$$\Theta_{\lambda,k}(q) = \sum_{n\in\mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}},$$

the $(\partial\Theta)_{\lambda,k}$ are the affine theta functions

$$(\partial\Theta)_{\lambda,k}(q) = \sum_{n\in\mathbb{Z}}(2kn+\lambda)q^{\frac{(2kn+\lambda)^2}{4k}}$$

and the $\chi_{M(h,c)}$ are the characters of the irreducible $\mathcal{L}$-modules with lowest weight h and central charge c (c.f. corollary 6.6). The theta functions, the affine theta functions and the eta function transform under the action of the modular group $SL(2,\mathbb{Z})$ represented by $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$ as

$$(S\eta)(\tau) = \sqrt{-i\tau}\eta(\tau) \tag{59a}$$

$$(T\eta)(\tau) = e^{\frac{i\pi}{12}}\eta(\tau) \tag{59b}$$

$$(S\Theta_{\lambda,k})(\tau) = \sqrt{\frac{-i\tau}{2k}}\sum_{\lambda'=0}^{2k-1} e^{i\pi\frac{\lambda\lambda'}{k}}\Theta_{\lambda',k}(\tau) \tag{59c}$$

$$(T\Theta_{\lambda,k})(\tau) = e^{i\pi\frac{\lambda^2}{2k}}\Theta_{\lambda,k}(\tau) \tag{59d}$$

$$(S(\partial\Theta)_{\lambda,k})(\tau) = \sqrt{\frac{1}{2k}}(-i\tau)^{\frac{3}{2}}\sum_{\lambda'=1}^{2k-1} e^{i\pi\frac{\lambda\lambda'}{k}}(\partial\Theta)_{\lambda',k}(\tau) \tag{59e}$$

$$(T(\partial\Theta)_{\lambda,k})(\tau) = e^{i\pi\frac{\lambda^2}{2k}}(\partial\Theta)_{\lambda,k}(\tau). \tag{59f}$$

Using the results of the preceding subsections we readily compute the characters

$$\chi_{\tilde{V}_0^{\mathcal{W}}}(q) = \chi_{V_0^{\mathcal{W}}}(q) + 2\chi_{V_1^{\mathcal{W}}}(q) = \frac{1}{2\eta(q)}(3\Theta_{1,2}(q) - (\partial\Theta)_{1,2}(q)), \tag{60a}$$

$$\chi_{V_0^{\mathcal{W}*}}(q) = \chi_{\tilde{V}_0^{\mathcal{W}}}(q) + \chi_{V_0^{\mathcal{W}}}(q) = \frac{2}{\eta(q)}\Theta_{1,2}(q), \tag{60b}$$

$$\chi_{V_1^{\mathcal{W}*}}(q) = \Big(2\chi_{V_0^{\mathcal{W}}}(q) + \chi_{V_1^{\mathcal{W}}}(q)\Big) + \chi_{V_1^{\mathcal{W}}}(q) = \frac{2}{\eta(q)}\Theta_{1,2}(q) = \chi_{\tilde{V}_0^{\mathcal{W}*}}(q). \tag{60c}$$

The characters $\chi_{V_0^{\mathcal{W}}}$ and $\chi_{V_1^{\mathcal{W}}}$ generate "character" functions with logarithmic terms in q under the action of the modular group, which we cannot interpret as characters of $\mathcal{W}(2,3^3)$-modules (in [22] it was attempted to interpret these functions in terms of generalized characters). Anyway, with respect to theorem 8.13 it seem to be more natural to view the $\mathcal{W}(2,3^3)$-modules

$$\Big\{V_0^{\mathcal{W}*}, V_1^{\mathcal{W}*}, V_{-1/8}^{\mathcal{W}}, V_{3/8}^{\mathcal{W}}\Big\} \tag{61}$$

as the building blocks of our theory. In fact, the characters of these modules display a much more agreeable behaviour under the action of the modular group. With (59) one calculates:

$$S\chi_{V_0^{\mathcal{W}*}} = \chi_{V_{-1/8}^{\mathcal{W}}} - \chi_{V_{3/8}^{\mathcal{W}}} \tag{62a}$$

$$S\chi_{V_1^{\mathcal{W}*}} = \chi_{V_{-1/8}^{\mathcal{W}}} - \chi_{V_{3/8}^{\mathcal{W}}} \tag{62b}$$

$$S\chi_{V_{-1/8}^{\mathcal{W}}} = \frac{1}{2}\chi_{V_{-1/8}^{\mathcal{W}}} + \frac{1}{2}\chi_{V_{3/8}^{\mathcal{W}}} + \frac{1}{2}\chi_{V_0^{\mathcal{W}*}} \tag{62c}$$

$$S\chi_{V_{3/8}^{\mathcal{W}}} = \frac{1}{2}\chi_{V_{-1/8}^{\mathcal{W}}} + \frac{1}{2}\chi_{V_{3/8}^{\mathcal{W}}} - \frac{1}{2}\chi_{V_0^{\mathcal{W}*}} \tag{62d}$$

$$T\chi_{V_0^{\mathcal{W}*}} = e^{\frac{i\pi}{6}}\chi_{V_0^{\mathcal{W}*}} \tag{62e}$$

$$T\chi_{V_1^{\mathcal{W}*}} = e^{\frac{i\pi}{6}}\chi_{V_1^{\mathcal{W}*}} \tag{62f}$$

$$T\chi_{V_{-1/8}^{\mathcal{W}}} = e^{-\frac{i\pi}{12}}\chi_{V_{-1/8}^{\mathcal{W}}} \tag{62g}$$

$$T\chi_{V_{3/8}^{\mathcal{W}}} = -e^{-\frac{i\pi}{12}}\chi_{V_{3/8}^{\mathcal{W}}}. \tag{62h}$$

Because of $\chi_{V_0^{\mathcal{W}*}} = \chi_{V_1^{\mathcal{W}*}}$, the definition of the modular matrices is somewhat arbitrary. With the base (61) the most general ansatz is given by

$$S = \begin{pmatrix} \alpha & \beta & \frac{\gamma}{2} & -\frac{\delta}{2} \\ -\alpha & -\beta & \frac{1-\gamma}{2} & \frac{\delta-1}{2} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; T = \begin{pmatrix} \mu e^{\frac{i\pi}{6}} & (1-\nu)e^{\frac{i\pi}{6}} & \lambda & \kappa \\ (1-\mu)e^{\frac{i\pi}{6}} & \nu e^{\frac{i\pi}{6}} & -\lambda & -\kappa \\ 0 & 0 & e^{-\frac{i\pi}{12}} & 0 \\ 0 & 0 & 0 & -e^{-\frac{i\pi}{12}} \end{pmatrix}. \tag{63}$$

There are six solutions satisfying $S^4 = \mathbb{1}$, $(ST)^3 = S^2$ and the charge conjugation matrix $C = S^2$ being a permutation matrix:

$$S_1 = \begin{pmatrix} \frac{i}{2} & -\frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{i}{2} & \frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; T_1 = \begin{pmatrix} 0 & e^{i\pi/6} & 0 & 0 \\ e^{i\pi/6} & 0 & 0 & 0 \\ 0 & 0 & e^{-i\pi/12} & 0 \\ 0 & 0 & 0 & -e^{-i\pi/12} \end{pmatrix}$$

$$S_2 = \begin{pmatrix} \frac{i}{2} & -\frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{i}{2} & \frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; T_2 = \begin{pmatrix} \frac{3+i\sqrt{3}}{4}e^{i\pi/6} & \frac{1-i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ \frac{1-i\sqrt{3}}{4}e^{i\pi/6} & \frac{3+i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ 0 & 0 & e^{-i\pi/12} & 0 \\ 0 & 0 & 0 & -e^{-i\pi/12} \end{pmatrix}$$

$$S_3 = \begin{pmatrix} \frac{i}{2} & -\frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{i}{2} & \frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; T_3 = \begin{pmatrix} \frac{3-i\sqrt{3}}{4}e^{i\pi/6} & \frac{1+i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ \frac{1+i\sqrt{3}}{4}e^{i\pi/6} & \frac{3-i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ 0 & 0 & e^{-i\pi/12} & 0 \\ 0 & 0 & 0 & -e^{-i\pi/12} \end{pmatrix}$$

$$S_4 = \begin{pmatrix} -\frac{i}{2} & \frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ \frac{i}{2} & -\frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; T_4 = \begin{pmatrix} e^{i\pi/6} & 0 & 0 & 0 \\ 0 & e^{i\pi/6} & 0 & 0 \\ 0 & 0 & e^{-i\pi/12} & 0 \\ 0 & 0 & 0 & -e^{-i\pi/12} \end{pmatrix}$$

$$S_5 = \begin{pmatrix} -\frac{i}{2} & \frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ \frac{i}{2} & -\frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; T_5 = \begin{pmatrix} \frac{1+i\sqrt{3}}{4}e^{i\pi/6} & \frac{3-i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ \frac{3-i\sqrt{3}}{4}e^{i\pi/6} & \frac{1+i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ 0 & 0 & e^{-i\pi/12} & 0 \\ 0 & 0 & 0 & -e^{-i\pi/12} \end{pmatrix}$$

$$S_6 = \begin{pmatrix} -\frac{i}{2} & \frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ \frac{i}{2} & -\frac{i}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \; T_6 = \begin{pmatrix} \frac{1-i\sqrt{3}}{4}e^{i\pi/6} & \frac{3+i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ \frac{3+i\sqrt{3}}{4}e^{i\pi/6} & \frac{1-i\sqrt{3}}{4}e^{i\pi/6} & 0 & 0 \\ 0 & 0 & e^{-i\pi/12} & 0 \\ 0 & 0 & 0 & -e^{-i\pi/12} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note, that cases (1)–(3) only differ in the T-matrix and cases (4)–(6) are just cases (1)–(3) with the roles of $V_0^{\mathcal{W}*}$ and $V_1^{\mathcal{W}*}$ interchanged.

If all characters were linearly independent, invariance of the partition function

$$Z = \sum_{i,j} m_{ij} \bar{\chi}_i \chi_j, m_{ij} \in \mathbb{N}^0,$$

would force S^t to be unitary with respect to the scalar product given by the matrix $M := (m_{ij})_{i,j}$, i.e. $\bar{S} M S^t = M$. Since (in the base (61)) χ_1 and χ_2 are the same, we have a weaker restriction:

$$\begin{aligned}
\sum_{i,j=1,2} (\bar{S} M S^t)_{i,j} &= \sum_{i,j=1,2} (M)_{i,j} && \text{(64a)} \\
\sum_{i=1,2} (\bar{S} M S^t)_{i,j} &= \sum_{i=1,2} (M)_{i,j}, \ j = 3,4 && \text{(64b)} \\
\sum_{j=1,2} (\bar{S} M S^t)_{i,j} &= \sum_{j=1,2} (M)_{i,j}, \ i = 3,4 && \text{(64c)} \\
(\bar{S} M S^t)_{i,j} &= (M)_{i,j}, \ i,j = 3,4. && \text{(64d)} \\
& && \text{(64e)}
\end{aligned}$$

In all six cases this, together with $m_{ij} \in \mathbb{N}^0$, fixes M to be of the form

$$M = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 2(\alpha + 2\beta + \gamma) & 0 \\ 0 & 0 & 0 & 2(\alpha + 2\beta + \gamma) \end{pmatrix}, \ \alpha, \beta, \gamma \in \mathbb{N}^0,$$

yielding the (up to a scalar factor) unique partition function

$$Z = 2(\alpha + 2\beta + \gamma) \left(2|\Theta_{1,2}|^2 + |\Theta_{0,2}|^2 + |\Theta_{2,2}|^2 \right),$$

which is the partition function of the Gaussian model with central charge $c = 1$ compactified on a circle of radius 1 (for some speculations on the relations between nonunitary and unitary CFTs with the same partition functions c.f. [23]).

8.8 Fusion rules

The modules (61) do indeed close under fusion, as was shown by M.R. Gaberdiel and H.G. Kausch [28]. In the base (61) the fusion matrices $(N_i)_{j,k}$ ($V_i \star V_j = \bigoplus_k (N_i)_{k,j} V_k$) are given by

$$\begin{aligned}
N_1 = N_2 &= \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}, && \text{(65a)} \\
N_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}, && \text{(65b)} \\
N_4 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}. && \text{(65c)}
\end{aligned}$$

It was conjectured by E. Verlinde [42] that for any rational model the S-matrix diagonalizes the fusion matrices

$$S^{-1} N_i^t S = D_i$$

where D_i is a diagonal matrix and

$$(D_i)_{jj} = \frac{S_{ij}}{S_{0,j}}$$

('0' is the index of the vacuum module). Unfortunately, in our model *neither* of the two different possible S-matrices diagonalizes the fusion rules, but transforms them into block-diagonal form. This was to be expected, since the vacuum module (with trivial fusion rules) only occurs as a *submodule* of one of our basic modules (61).

The quantum dimensions

$$d(V) := \lim_{i\tau \nearrow 0} \frac{\chi_V(\tau)}{\chi_{V_0^{\mathcal{W}}}(\tau)}, \tag{66}$$

can be calculated via

$$d(V_3) = \lim_{q \nearrow 1} \frac{\Theta_{0,2}(q)}{\Theta_{1,2}(q) + (\partial\Theta)_{1,2}(q)} = \lim_{q \searrow 0} \frac{2(\Theta_{0,2}(q) - 2\Theta_{1,2}(q) + \Theta_{2,2}(q))}{\frac{1}{i\pi}\log q(\partial\Theta)_{1,2}(q) + \Theta_{0,2}(q) - \Theta_{2,2}(q))} = 2 \tag{67}$$

and

$$\frac{d(V_i)}{d(V_3)} = \lim_{i\tau \nearrow 0} \frac{\sum_{j=1}^4 S_{j,i}\chi_{V_j}(-\frac{1}{\tau})}{\sum_{k=1}^4 S_{k,3}\chi_k(-\frac{1}{\tau})} = \lim_{q \to 0} \frac{\sum_{j=1}^4 S_{j,i}\chi_j(q)}{\sum_{k=1}^4 S_{k,3}\chi_k(q)} = \frac{S_{3,i}}{S_{3,3}}. \tag{68}$$

They are given by

$$d(V_0^{\mathcal{W}*}) = d(V_1^{\mathcal{W}*}) = 4,\ d(V_{-1/8}^{\mathcal{W}}) = d(V_{3/8}^{\mathcal{W}}) = 2. \tag{69}$$

As expected, they indeed transform multiplicatively under fusion.

9 Conclusions and outlook

As we have seen, many of the properties of arbitrary representations of the Virasoro algebra can be deduced from lowest weight representations. In particular, there are no new critical exponents which do not occur in lowest weight modules. For the (simple) case of Jordan Verma modules, their maximal preserving submodules were determined, yielding a formula for the characters of minimal Jordan lowest weight representations.

For general staggered modules, we found strong restrictions on their submodules and proved the moduli spaces $\mathcal{V}_{h,h'}$ to be one-dimensional vector spaces if there is an embedding $V(h',c) \to V(h,c)$, and to be empty otherwise.

It remains an open question, whether it is possible to embed a Jordan Verma module into another as a proper submodule of the maximal preserving submodule. Connected to that, the classification of the maximal preserving submodule of a staggered modules with given lowest weights h^1, h^2 also is an open problem (its maximal *proper* submodule is, of course, either a Jordan Verma modules ($\alpha = 0$) or itself a staggered submodule with lowest weights h^2 or h^3, c.f. section 5.1).

It is another problem, to extend the results on the moduli space of staggered modules to higher ranks (e.g. in the rank 3 case, $L_0^n v^{(2)}$ is *not* necessarily singular, and therefore the moduli space $\mathcal{V}_{h^1,h^2,h^3}$ is not the direct sum of $\mathcal{V}_{h^1,h^2}$ and $\mathcal{V}_{h^2,h^3}$). In addition, the structure of maximal preserving submodules of Jordan Verma modules can be more complicated at higher rank. At $c = -2, h = \frac{3}{8}$, e.g., the maximal preserving submodule of a rank 3 JVM is a staggered module with lowest weights $\frac{35}{8}$, $\frac{35}{8}$ and $\frac{99}{8}$, which is neither a JLWM nor a strictly staggered module.

The representation theory of $\mathcal{W}$-algebras in the logarithmic regime was exemplarily studied for the case of $\mathcal{W}(2,3^3)$ at $c = -2$ yielding finitely many representations from which all others can be constructed by taking submodules, factor modules and direct sums. These basic representations

close under fusion and their characters span a finite dimensional representation of the modular group.
Various reasons suggest that similar results will hold for the whole series of triplet algebras $\mathcal{W}(2,(2p-1)^3)$ at $c=c_{1,p}$, but this is yet to be proven.

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A Notations

The modes of a field Φ with conformal weight h_Φ are defined by

$$\Phi(z)=\sum_{n=-\infty}^{\infty}\Phi_n z^{n-h_\Phi}.$$

For two fields Φ,Ψ with conformal weights h_Φ,h_Ψ their normal ordered product is defined as

$$N(\Phi,\Psi):=\sum_{n=-\infty}^{\infty}N(\Phi,\Psi)_n z^{n-(h_\Phi+h_\Psi)}$$

with

$$N(\Phi,\Psi)_n:=\varepsilon_{\Phi\Psi}\sum_{k=-\infty}^{h_\Psi-1}\Phi_{n-k}\Psi_k+\sum_{k=h_\Psi}^{\infty}\Psi_k\Phi_{n-k},$$

where $\varepsilon_{\Phi\Psi}=-1$ if both Φ and Ψ are fermionic and $+1$ in all other cases.
The quasiprimary normal ordered product $\mathcal{N}(\Phi,\Psi)$ is the projection of $N(\Phi,\Psi)$ onto the space of quasiprimary fields. If $\{\Phi_i, i\in I\}$ is a base of the space of quasiprimary fields, it is explicitly given by the formula

$$\begin{aligned}\mathcal{N}(\Phi_j,\partial^n\Phi_i) &= \sum_{r=0}^{n}(-)^r\binom{n}{r}\binom{2(h_i+h_j+n-1)}{r}^{-1}\binom{2h_i+n-1}{r}\partial^r N^{(h_i+n-r)}(\Phi_j,\partial^{n-r}\Phi_i)\\ &\quad -(-)^n\sum_{\{k|h_{ijk}\geq 1\}}C_{ij}^k\binom{h_{ijk}+n-1}{n}\binom{2(h_i+h_j+n-1)}{n}^{-1}\\ &\quad\times\binom{2h_i+n-1}{h_{ijk}+n}\binom{\sigma_{ijk}-1}{h_{ijk}-1}^{-1}\frac{\partial^{h_{ijk}+n}\Phi_k}{(\sigma_{ijk}+n)(h_{ijk}-1)!}\end{aligned}$$

where $\sigma_{ijk}:=h_i+h_j+h_k-1$, $h_{ijk}=h_i+h_j-h_k$ and the C_{ij}^k are the structure constants of the chiral algebra.
With the above notations the isomorphism between the fields and the vacuum representation is given by

$$\begin{aligned}\rho\left(\Phi\right) &= \Phi_{h_\Phi}|0\rangle,\\ \rho^{-1}\left(\Phi_{i_1,n_1}\dots\Phi_{i_k,n_k}|0\rangle\right) &= N(N(\dots N(\Phi_{i_k}^{(n_k)},\Phi_{i_{k-1}}^{(n_{k-1})}),\dots),\Phi_{i_1}^{(n_1)})\end{aligned}$$

where h_i is the conformal weight of $\Phi^{(i)}$, $\Phi_i^{(n)}:=\frac{\partial^{n-h_i}}{(n-h_i)!}\Phi_i$ and $\forall k: n_i\geq h_i$.

For more details see e.g. [5].

References

[1] A. Belavin, A. Polyakov, A. Zamolodchikov,
"Infinite conformal symmetry in two-dimensional quantum field theory",
Nucl. Phys. B241 (1984) 333

[2] A. Bilal, I.I. Kogan,
"On Gravitational Dressing of 2D Field Theories in Chiral Gauge",
Nucl. Phys. B449 (1995) 569

[3] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, R. Hübel,
"Unifying $\mathcal{W}$-Algebras",
Phys. Lett. B332 (1994) 51

[4] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, R. Hübel,
"Coset Realization of Unifying $\mathcal{W}$-Algebras",
Int. J. Mod. Phys. A10 (1995) 2367

[5] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel, R. Varnhagen,
"$\mathcal{W}$-algebras with two and three generators",
Nucl. Phys. B361 (1991) 255

[6] R. Blumenhagen, A. Wisskirchen,
"Extension of the $N = 2$ Virasoro algebra by two primary fields of dimension 2 and 3",
Phys. Lett. B343 (1995) 168

[7] J.L. Cardy, A.J. Guttmann,
"Universal Amplitude Combinations for Self-Avoiding Walks, Polygons and Trails",
cond-mat/9303035

[8] J.-S. Caux, I.I. Kogan, A.M. Tsvelik,
"Logarithmic Operators and Hidden Continuus Symmetry in Critical Disordered Models",
preprint OUTP-95-62 (1995), hep-th/9511134

[9] B. Duplantier, H. Saleur,
"Exact Critical Properties of Two-Dimensional Dense Self-Avoiding Walks",
Nucl. Phys. B290 (1987) 291

[10] W. Eholzer, N.-P. Skoruppa,
"Modular Invariance and Uniqueness of Conformal Characters",
Commun. Math. Phys. 174 (1995) 117

[11] J. Ellis, N.E. Mavromatos, D.V. Nanopoulos,
"D-Branes from Liouville Strings",
preprint ACT-04/96, CERN-TH/96-81, CTP-TAMU-11/96, OUTP-96-15P,
hep-th/9605046

[12] J. Ellis, N.E. Mavromatos, D.V. Nanopoulos,
"D-Brane Recoil Mislays Information",
preprint ACT-14/96, CERN-TH/96-264, CTP-TAMU-49/96, OUTP-96-57P,
hep-th/9609238

[13] B.L. Feigin, D.B. Fuks,
"Invariant skew-symmetric differential operators on the line and the Verma modules over the Virasoro algebra",
Funct. Anal. Appl. 16 (1982) 114

[14] B.L. Feigin, D.B. Fuks,
"Verma Modules over the Virasoro Algebra",
Funct. Anal. Appl. 17 (1983) 241

[15] B.L. Feigin, D.B. Fuks,
"Verma Modules over the Virasoro Algebra",
in "Topology", Proc. Leningrad 1982, L.D. Faadev, A.A.Mal'cev (eds.), Lecture Notes Math. 1060, Springer-Verlag (1984) 230

[16] B.L. Feigin, D.B. Fuks,
"Casimir Operators in Modules over the Virasoro Algebra",
Soviet Math. Dokl. Vol. 27, No. 2 (1983) 465

[17] B.L. Feigin, T. Nakanishi, H. Ooguri,
"The Annihilating Ideals of Minimal Models",
Int. J. Mod. Phys. A7 (1992) 217

[18] G. Felder,
"BRST Approach to Minimal Models",
Nucl. Phys. B317 (1989) 215,
Erratum, Nucl. Phys. B324 (1989) 548

[19] G. Felder, J. Fröhlich, G. Keller,
"Braid Matrices and Structure Constants for Minimal Conformal Models",
Commun. Math. Phys. 124 (1989) 647

[20] M. Flohr,
"W-Algebras, New Rational Models and Completeness of the c=1 Classification",
Commun. Math. Phys. 157 (1993) 179

[21] M. Flohr,
"Die rationalen konformen Quantenfeldtheorien in zwei Dimensionen mit effektiver zentraler Ladung $c_{\text{eff}} \leq 1$",
BONN-IR-94-11 (Ph.D. thesis, in german) (1994)

[22] M. Flohr,
"On Modular Invariant Partition Functions of Conformal Field Theories with Logarithmic Operators",
Int. J. Mod. Phys. A11 (1996) 4147

[23] M. Flohr,
"On Fusion Rules in Logarithmic Conformal Field Theories",
preprint IASSNS-HEP-96-54, hep-th/9605151 (1996),
to be published in Int. Jour. Mod. Phys. A

[24] M. Flohr,
"Fusion and Tensoring of Conformal Field Theories and Composite Fermion Picture of Fractional Quatum Hall Effect",
Mod. Phys. Lett. A11 (1996) 55-68

[25] M. Flohr,
"2-Dimensional Turbulence: yet another Conformal Field Theory Solution",
preprint IASSNS-HEP-96-69, hep-th/9606130 (1996), to be published in Nucl. Phys. B

[26] M.R. Gaberdiel,
"Fusion in conformal field theory as the tensor product of the symmetry algebra",
Int. J. Mod. Phys. A9 (1994) 4636

[27] M.R. Gaberdiel, H.G. Kausch,
"Indecomposable Fusion Products",
hep-th/9604026

[28] M.R. Gaberdiel, H.G. Kausch,
"A Rational Logarithmic Conformal Field Theory",
hep-th/9606050

[29] V. Gurarie,
"Logarithmic Operators in Conformal Field Theory",
Nucl. Phys. B410 (1993) 535

[30] J.E. Humphreys,
"Introduction to Lie Algebras and Representation Theory",
Springer-Verlag 1972

[31] V.G. Kac,
"Contravariant Form for the Infinite-Dimensional Lie-Algebras and Superalgebras",
Lecture Notes in Physics 94, Springer-Verlag (1979) 441

[32] H.G. Kausch,
"Extended Conformal Algebras Generated by a Multiplet of Primary Fields",
Phys. Lett. B259 (1991) 448

[33] H.G. Kausch,
"Curiosities at $c = -2$",
preprint DAMTP 95-52 , hep-th/9510149

[34] I.I. Kogan, N.E. Mavromatos,
"World-Sheet Logarithmic Operators and Target Space Symmetries in String Theory",
preprint OUTP-95-50P (1995), hep-th/9512210

[35] Z. Maassarani, D. Serban, "Non-Unitary Conformal Field Theory and Logarithmic Operators for Disordered Systems",
preprint SPHT-T96/037 (1996), hep-th/9605062

[36] M. Milovanović, N. Read,
"Edge exitations of paired fractional quantum Hall states",
preprint Yale Univ. (1996), cond-mat/9602113

[37] G. Moore, N. Seiberg,
"Classical and Quantum Conformal Field Theory",
Commun. Math. Phys. 123 (1989) 177

[38] W. Nahm,
"Quasi-rational fusion products",
Int. J. Mod. Phys. B8 (1994) 3693

[39] F. Rohsiepe,
"Nichtunitäre Darstellungen der Virasoro-Algebra mit nichttrivialen Jordanblöcken",
BONN-IB-96-19 (diploma thesis, in german) (1996)

[40] L. Rozansky, H. Saleur,
"S and T Matrices for the Super U(1,1) WZW Model",
Nucl. Phys. B389 (1993) 365

[41] H. Saleur,
"Polymers and percolation in two dimensions and twisted $N = 2$ supersymmetry",
Nucl. Phys. B382 (1992) 486

[42] E. Verlinde,
"Fusion Rules and Modular Transformations in 2d Conformal Field Theory",
Nucl. Phys. B300 (1988) 360

[43] X.-G. Wen, Y.-S. Wu, Y. Hatsugai,
"Chiral operator product algebra and edge excitations of a fractional quantum Hall droplet",
Nucl. Phys. B422 (1994) 476